Nucleon structure functions from lattice operator product expansion

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QCDSF Collaboration

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Classical Approach

- Moments

$$\mu_n(q^2) = f \int_0^1 dx\, x^n\, F_1(x, q^2)$$

Mellin transform

$$F_1(x, q^2) = \frac{1}{2\pi i f} \int_{c-i\infty}^{c+i\infty} ds\, x^{-s-1} \mu_s(q^2)$$

- OPE

$$\mu_1(q^2) = c_2(q^2 a^2)\langle N|\mathcal{O}_2(a)|N\rangle + \frac{c_4(q^2 a^2)}{q^2}\langle N|\mathcal{O}_4(a)|N\rangle + \cdots$$

$$\mu_2(q^2) = c_3(q^2 a^2)\langle N|\mathcal{O}_3(a)|N\rangle + \frac{c_5(q^2 a^2)}{q^2}\langle N|\mathcal{O}_5(a)|N\rangle + \cdots$$

$$\vdots$$

- The computations are limited to a few lower moments, due to issues of operator mixing and renormalization. Even so, the uncertainties are at least comparable to the magnitude of the power corrections

Martinelli & Sachrajda

OPE without OPE

Mother of all: Compton amplitude

$$T_{\mu\nu}(p,q) = \rho_{\lambda\lambda'} \int \mathrm{d}^4x \, \mathrm{e}^{iqx} \langle p, \lambda' | T J_\mu(x) J_\nu(0) | p, \lambda \rangle$$

unpolarized: $2\rho = \mathbf{1}$

$$= \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \mathcal{F}_1(\omega, q^2) + \left(p_\mu - \frac{pq}{q^2} q_\mu\right) \left(p_\nu - \frac{pq}{q^2} q_\nu\right) \frac{1}{pq} \mathcal{F}_2(\omega, q^2)$$

For $p_3 = q_3 = q_4 = 0$

$$T_{33}(p,q) = \mathcal{F}_1(\omega, q^2) = 4\omega \int_0^1 dx \, \frac{\omega x}{1 - (\omega x)^2} F_1(x, q^2) + \mathcal{F}_1(0, q^2)$$

$$\omega = \frac{2pq}{q^2}$$

$$= \sum_{n=2,4,\cdots}^{\infty} 4\omega^n \int_0^1 dx \, x^{n-1} F_1(x, q^2) + \mathcal{F}_1(0, q^2)$$

includes power corrections

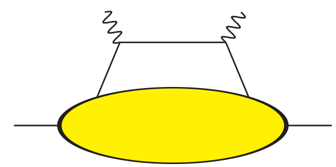
From T_{33} to μ_n and $F_1(x,q^2)$

The Compton amplitude can be computed most efficiently, including singlet (disconnected) matrix elements, by the Feynman-Hellmann technique. By introducing the perturbation to the Lagrangian

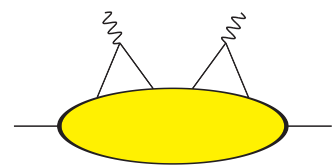
$$\mathcal{L}(x) \to \mathcal{L}(x) + \lambda \mathcal{J}_3(x)\,, \quad \mathcal{J}_3(x) = Z_V \cos(\vec{q}\vec{x})\; e_q\, \bar{q}(x)\gamma_3 q(x)$$

and taking the second derivative of $\langle N(\vec{p},t)\bar{N}(\vec{p},0)\rangle_\lambda \simeq C_\lambda\, \mathrm{e}^{-E_\lambda(p,q)\,t}$ with respect to λ on both sides, we obtain

$$-2E_\lambda(p,q)\,\frac{\partial^2}{\partial\lambda^2}E_\lambda(p,q)\,\big|_{\lambda=0} = T_{33}(p,q)$$



The amplitude encompasses the dominating 'handbag' diagram as well as the power-suppressed 'cats ears' diagram. Varying q^2 will allow to test the twist expansion. No further renormalization is needed



Moments

Task: Compute the lowest M moments [odd moments need $\langle p, \lambda' | T J_\mu(x) J^5_\nu(0) | p, \lambda \rangle$]

$$\mu_{2m-1} = \int_0^1 dx\, x^{2m-1} F_1(x)$$

from a finite number of sampled points

$$t_n = T_{33}(\omega_n)\,,\ \ n = 1, \cdots, N$$

Compton amplitude and moments are connected by the set of equations

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{pmatrix} = \begin{pmatrix} 4\omega_1^2 & 4\omega_1^4 & \cdots & 4\omega_1^{2M} \\ 4\omega_2^2 & 4\omega_2^4 & \cdots & 4\omega_2^{2M} \\ \vdots & \vdots & \vdots & \vdots \\ 4\omega_N^2 & 4\omega_N^4 & \cdots & 4\omega_N^{2M} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_3 \\ \vdots \\ \mu_{2M-1} \end{pmatrix}$$

Vandermonde M

Solutions are well documented in the literature. Alternatively, we can fit the Compton amplitude by the interpolating polynomial

$$T_{33}(\omega) = 4\left(\omega^2 \mu_1 + \omega^4 \mu_3 + \cdots + \omega^{2M} \mu_{2M-1}\right)$$

Structure function

Ultimate goal: Compute $F_1(x)$ from $T_{33}(\omega)$. Therefor we discretize the integral

$$t_n = \epsilon \sum_{m=1}^{M} K_{nm} f_m\,, \quad n = 1, \cdots, N$$

[here: points equidistant with step size ϵ]

with

$$f_m = F_1(x_m)\,, \quad K_{nm} = \frac{4\,\omega_n^2 x_m}{1-(\omega_n x_m)^2}\,, \quad N < M$$

The $N \times M$ matrix K is written

$$K = U\ [\mathrm{diag}(w_1, \cdots, w_N)]\ V^T$$

where W is singular: $w_k \approx 0, K < k \leq N$. Solution by singular value decomposition (SVD)

$$f_m = \sum_{n=1}^{N} K_{mn}^{-1} \epsilon^{-1}\, t_n$$

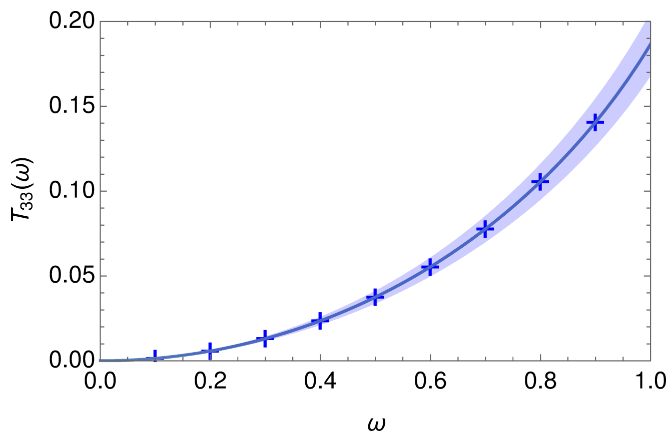
with K^{-1} being the pseudoinverse

$$K^{-1} = V\ [\mathrm{diag}(1/w_1, \cdots, 1/w_K, 0, \cdots, 0)]\ U^T$$

Mathematica

Proof of Concept

In

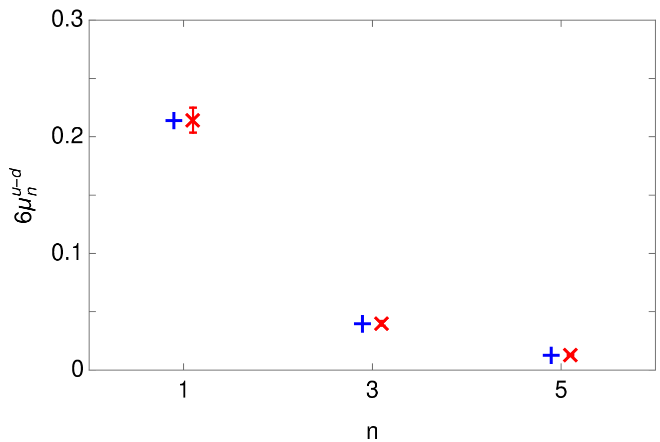


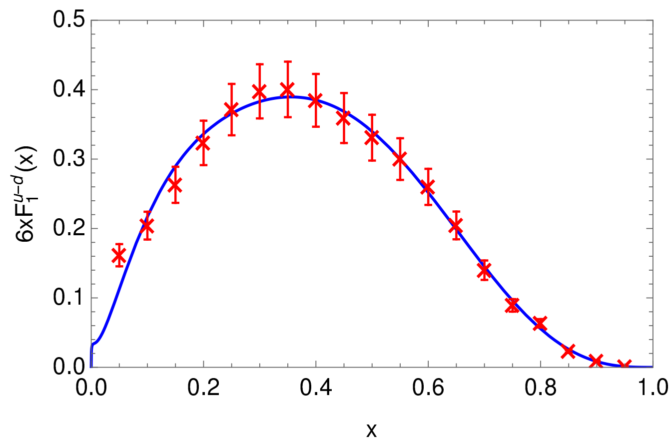
$$T_{33}(\omega) = 4\omega \int_0^1 dx \, \frac{\omega x}{1-(\omega x)^2} F_1^{u-d}(x)$$

$$2x F_1^{u-d}(x) = \frac{1}{3} x[u(x) - d(x)]$$

MSTW-lo

Out





$F_1(x)$ at very small x? Needs $\omega > 1$ Not accessible via moments

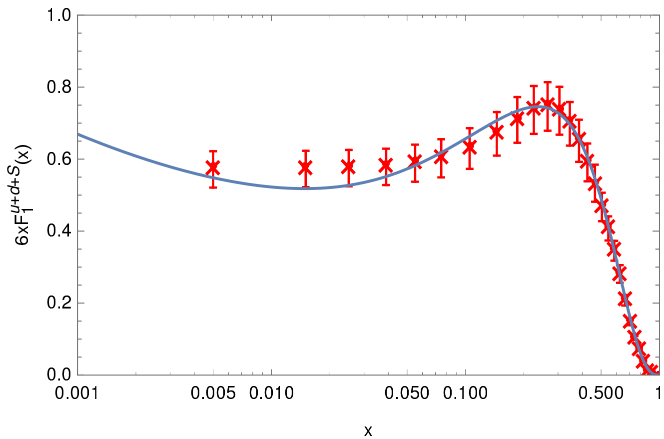
In

Out

$$T_{33}(\omega) = 4\omega \, \mathrm{P}\int_0^1 dx \, \frac{\omega x}{1-(\omega x)^2} F_1^{u+d+S}(x)$$

$$2xF_1^{u+d+S}(x) = \frac{1}{3}\, x[u(x) + d(x) + S(x)]$$

MSTW-lo

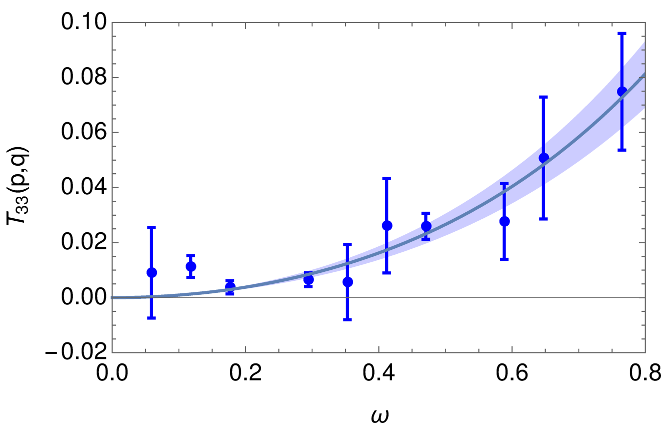


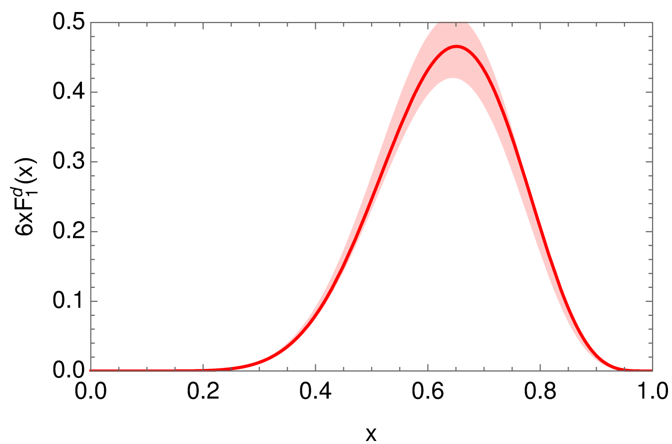
$\omega \in [0, 2]$

Lattice Study

SU(3) symmetric point

V	M_π	M_K	a [fm]	q^2 [GeV^2]
$32^3 \times 64$	420	420	0.075	9.2





$$\mathcal{J}_3(x) = Z_V \cos(\vec{q}\vec{x})\ e_d\,\bar{d}(x)\gamma_3 d(x)$$

Outlook

- Computations can be improved in many respects
  - Apply Bayesian regression with SVD to alleviate overfitting
  - Employ momentum smearing techniques for larger values of ω
- With gradual improvements, we should be able to compute the structure functions $F_1(x, q^2)$ and $F_2(x, q^2)$, as well as $g_1(x, q^2)$ and $g_2(x, q^2)$, including contributions of higher twist, from the Compton amplitude with unprecedented accuracy
- This is possible, because the calculation skirts the issue of renormalization and operator mixing
- The method can easily be generalized to generalized parton distribution functions (GPDs) $H(x, \xi, q^2)$ and $E(x, \xi, q^2)$