Lee-Yang zeros from canonical approach in the NJL model

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HaRP workshop “Hadrons and dense matter from QCD”
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1. **Introduction**
   Motivation, Sign problem

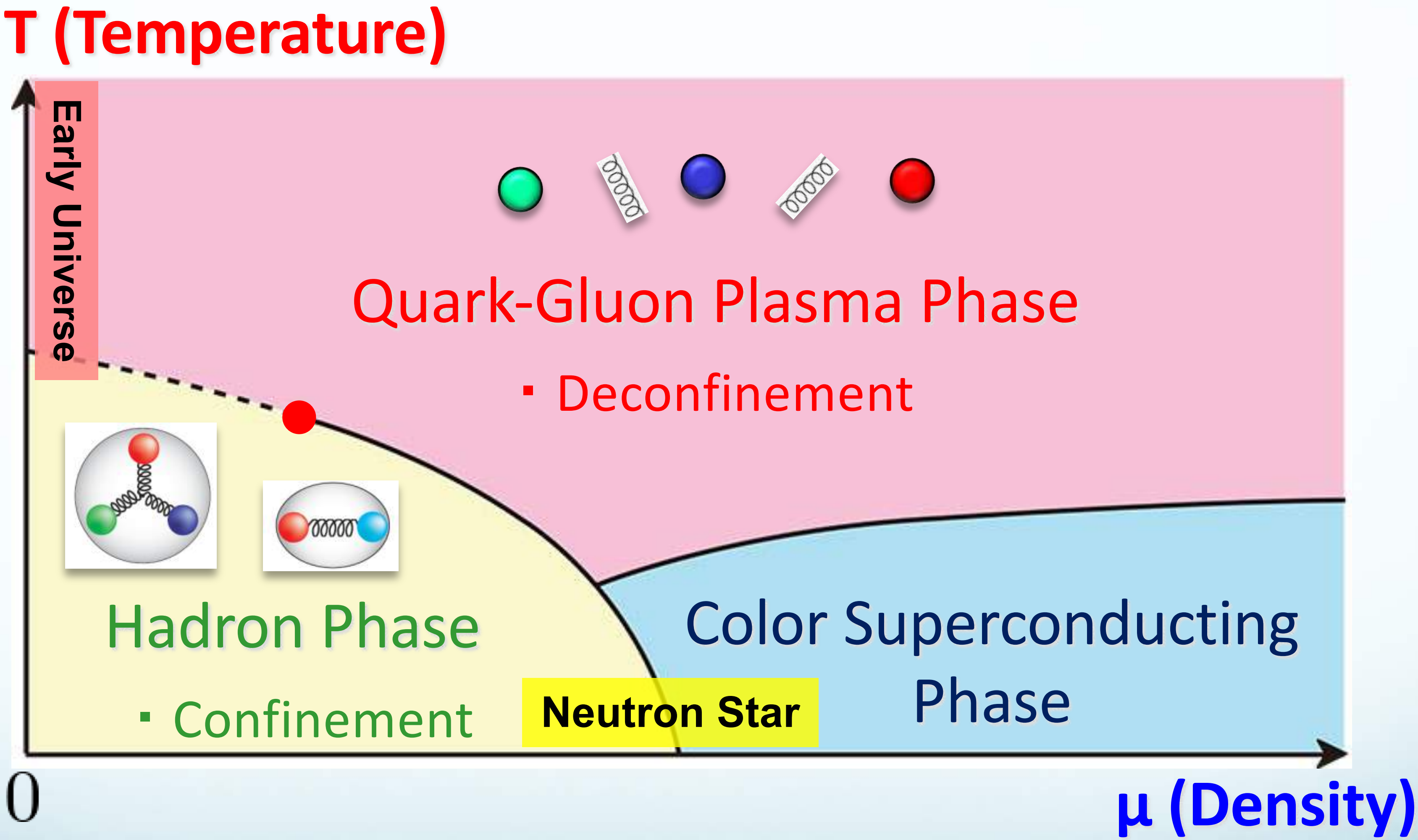
2. **Canonical approach and Lee-Yang zeros**
   Basic idea, History, Outline of the strategy

3. **Results of Lee-Yang zeros in lattice QCD**

4. **Results of Lee-Yang zeros in NJL model**
   volume dependence, fitting dependence,
   maximal number density dependence, extrapolation

5. **Summary & Future work**

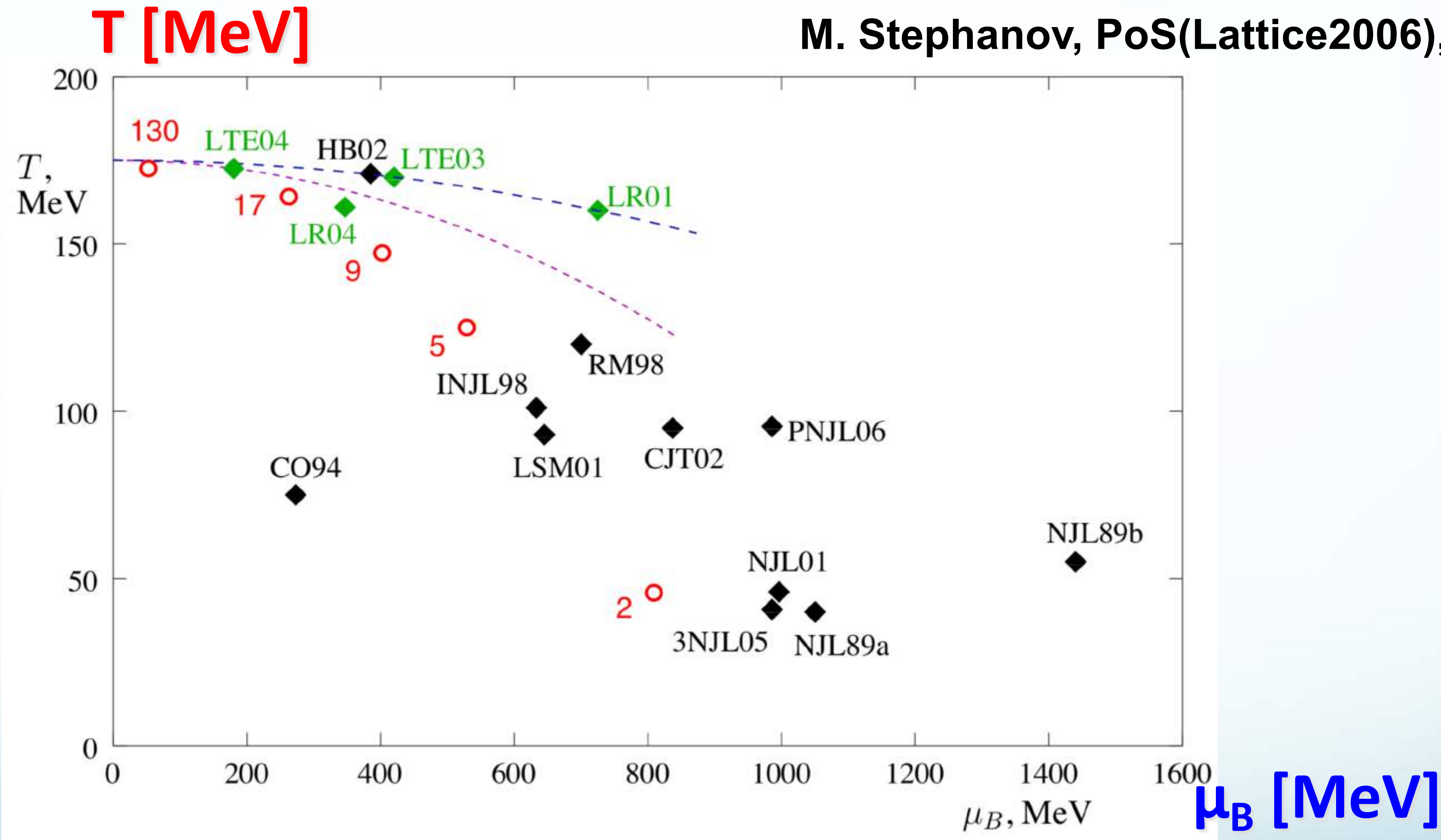
QCD Phase diagram (Prediction)



Where are the critical point and the phase transition lines?

Predicted critical points

M. Stephanov, PoS(Lattice2006), 024



Where are the critical point and the phase transition lines?
Lattice QCD at finite density: Existence of the sign problem

Sign Problem

S_G : Gauge action
$D(\mu_q)$: Fermion matrix
N_f : # of flavors

Monte Carlo Method

(importance sampling)

$$\langle \mathcal{O} \rangle_{\mu_q} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}[U_i]$$ **with Probability:** $$[\det D(\mu_q)]^{N_f} e^{-S_G}$$

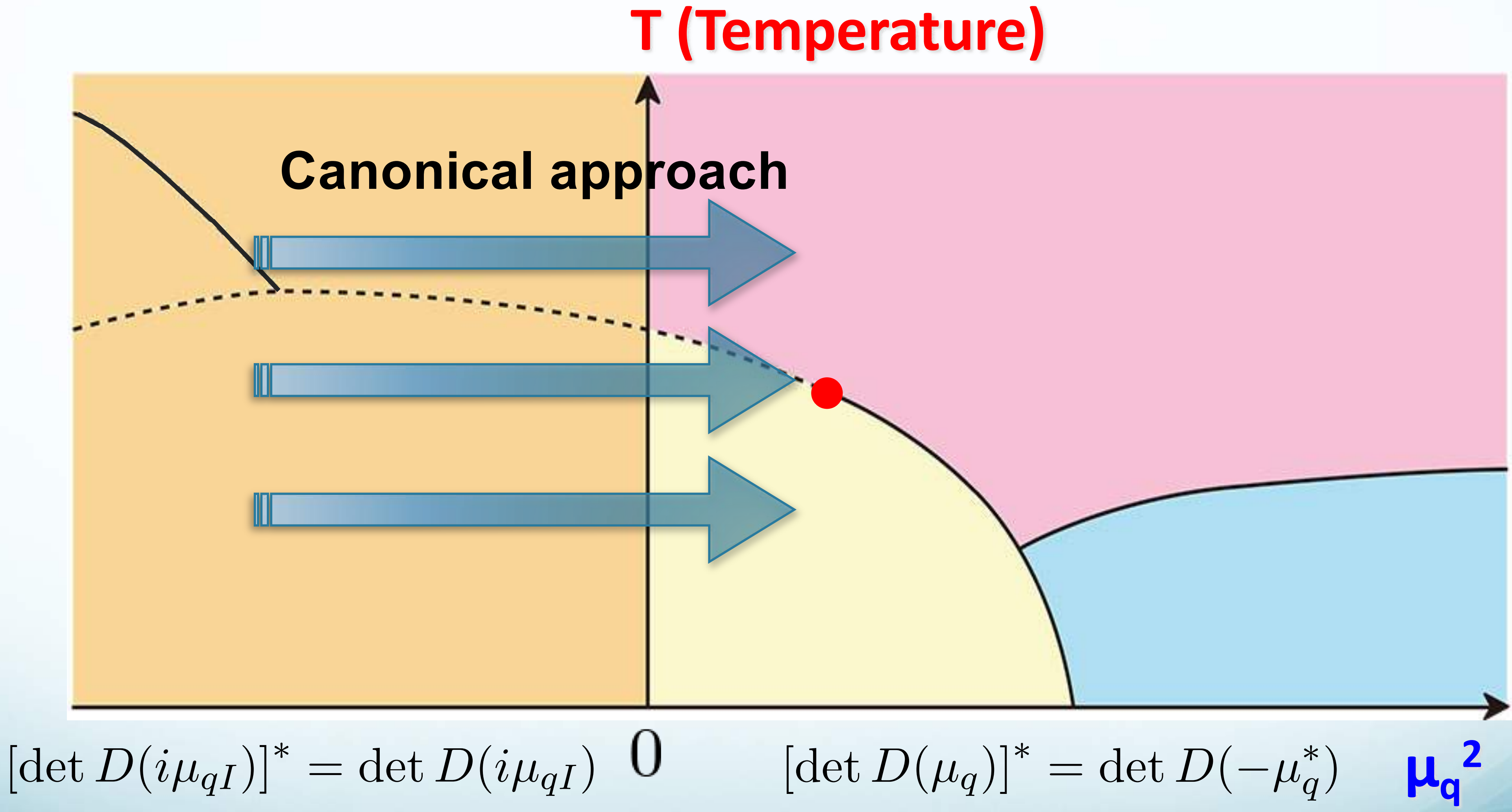
Chemical potential	$\det D(\mu_q)$	Monte Carlo Method
$\mu_q = 0$	Real value	○
$\mu_q \neq 0$	Complex value	✕ **(Sign Problem)**
Pure Imag. $\mu_q = i\mu_{qI}$	**Real value**	○

$$D(\mu_q) = D_\nu \gamma_\nu + m + \mu_q \gamma_0$$

$$D(\mu_q)^\dagger = -D_\nu \gamma_\nu + m + \mu_q^* \gamma_0 = \gamma_5 D(-\mu_q^*) \gamma_5$$

$$[\det D(\mu_q)]^* = \det \left[D(\mu_q)^\dagger \right] = \det \left[\gamma_5 D(-\mu_q^*) \gamma_5 \right] = \det D(-\mu_q^*)$$

QCD Phase diagram



Pure imaginary chemical potential: $\mu_q = i\mu_{qI}$

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Canonical Approach

Fugacity expansion

Grand Canonical partition function

$$
\begin{aligned}
\underline{Z_{\mathrm{GC}}(\mu_q, T, V)} &= \mathrm{Tr}\left(e^{-(\hat{H}-\mu_q \hat{N})/T}\right) \\
&= \sum_n \langle n| \, e^{-(\hat{H}-\mu_q \hat{N})/T} \, |n\rangle \\
&= \sum_n \langle n| \, e^{-\hat{H}/T} \, |n\rangle \, e^{n\mu_q/T} \\
&= \sum_n \underline{Z(n, T, V)}\xi^n
\end{aligned}
$$

Fugacity: $\xi = e^{\mu_q/T}$

Canonical partition function

Canonical Approach

Fugacity expansion

Grand Canonical partition function

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z(n, T, V)\xi^n$$

Fugacity: $\xi = e^{\mu_q/T}$

Canonical partition function

Fourier transformation

A. Hasenfrantz & D. Toussaint,
Nucl. Phys. B371 (1992)

$$Z(n, T, V) = \int_0^{2\pi} \frac{d(\mu_{qI}/T)}{2\pi} e^{-in\mu_{qI}/T} Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$$

We can calculate Z_{GC} with Monte Carlo Method at pure imaginary μ_q.

$$[\det D(i\mu_{qI})]^* = \det D(i\mu_{qI})$$

History

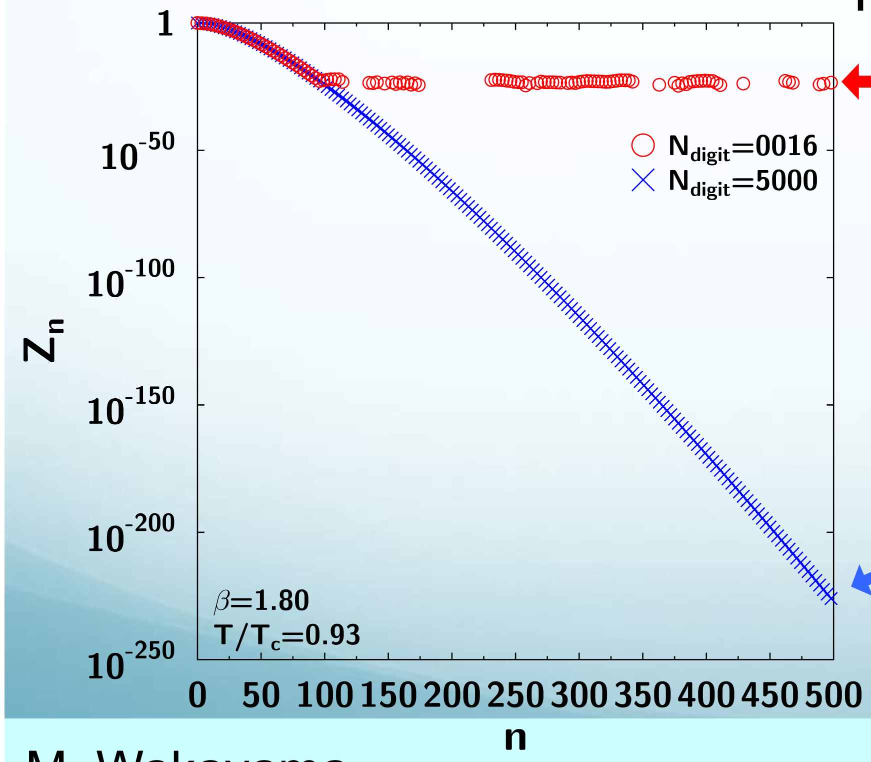
Basic Idea of Canonical Approach

A. Hasenfrantz, D. Toussaint, Nucl. Phys. B371 **(1992)**

❌ **Numerical instability of (discrete) Fourier transformation**

Sign Problem ? ⇒ No, this is caused by cancelation of significant digits !

R.Fukuda, A.Nakamura, S.Oka, PRD93 **(2016)**



In double-precision arithmetic, cancelation of significant digits occurs at high n region.

In multiple-precision arithmetic, we can evaluate Zn up to high n region with accuracy.

Outline of the strategy

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Number density formulation
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$$

Fourier transform

$$Z(n, T, V)$$

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

If we get Z_n for all n, we can search at ANY density!

Outline of the strategy

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Number density formulation
V. Bornyakov et al., PRD95(2017)

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Fourier transform

$$Z(n, T, V)$$

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

In numerical calculations, n is finite.

Lee-Yang Zeros

Zeros of Z_{GC} so-called Lee-Yang Zeros contain a valuable information on the phase transitions of a system.

T.D. Lee & C.N. Yang, Phys. Rev. 87, 404&410 (1952)

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n = 0$$

There are $2N_{max}$ LYZs
in the complex $\xi = e^{\mu_q/T}$ plane.

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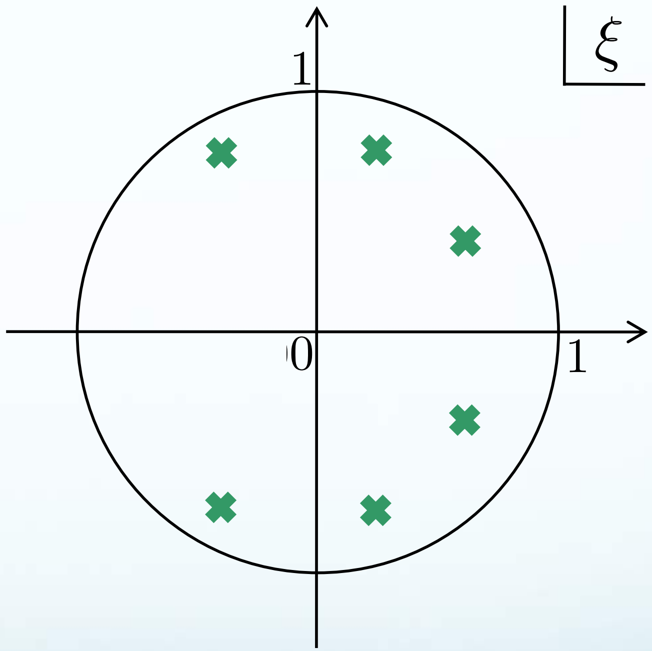

N_{max} ~ small

Lee-Yang Zeros

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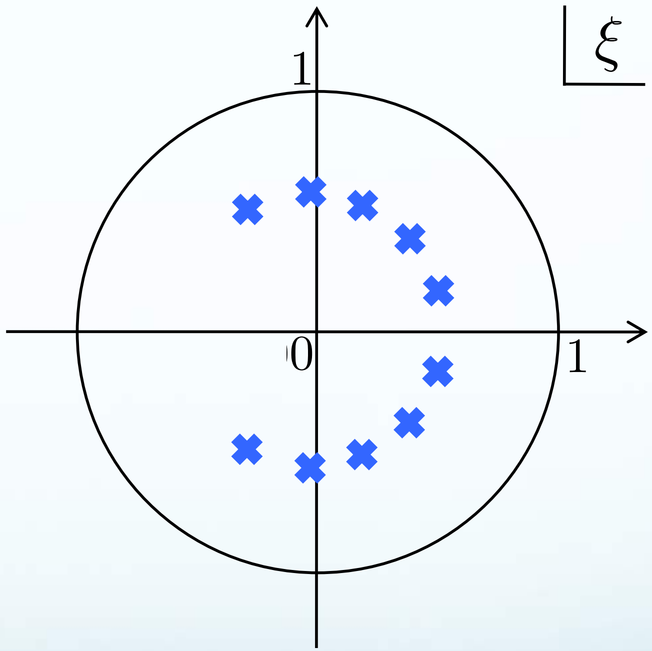

N_{max} ~ large

Lee-Yang Zeros

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T.D. Lee & C.N. Yang, Phys. Rev. 87, 404&410 (1952)

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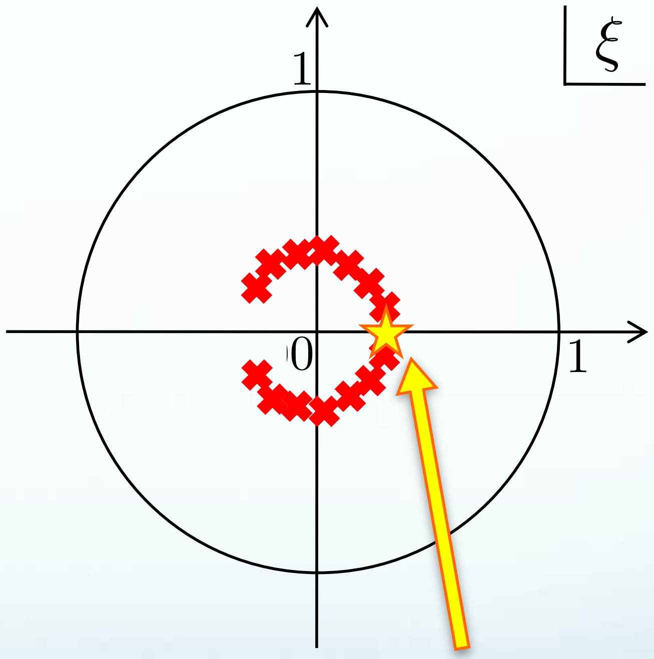
There are $2N_{max}$ LYZs in the complex $\xi = e^{\mu_q/T}$ plane.



Phase Transition
N_{max} ~ infinity
(V ~ infinity)

Lee-Yang Zeros

Zeros of Z_{GC} so-called Lee-Yang Zeros contain a valuable information on the phase transitions of a system.

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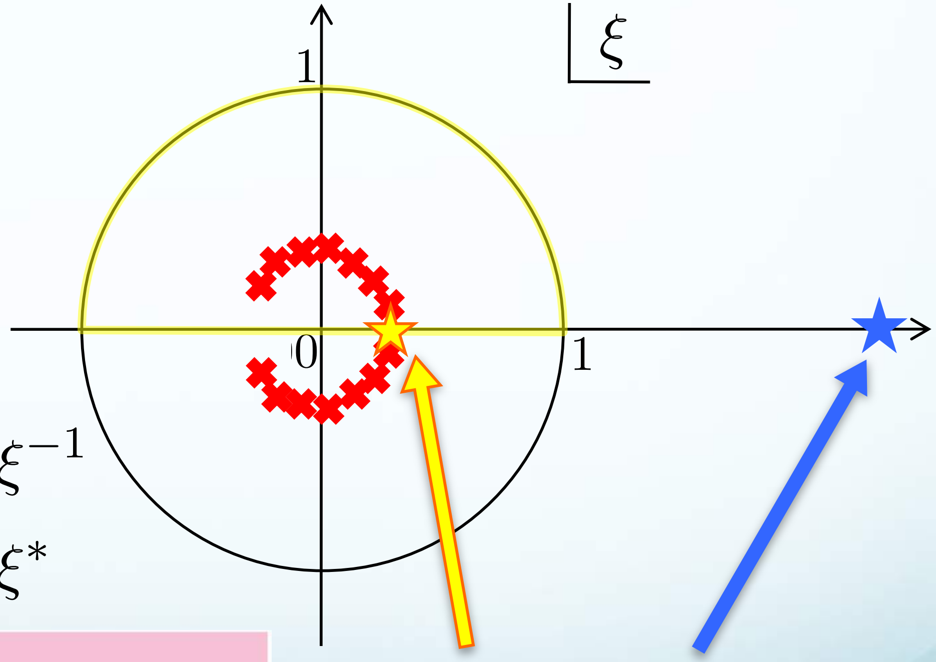


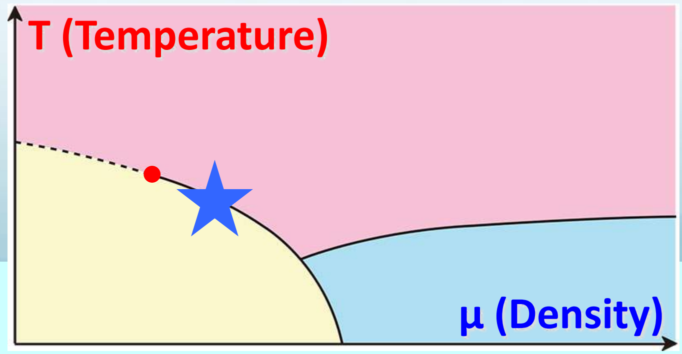
There are $2N_{max}$ LYZs in the complex $\xi = e^{\mu_q/T}$ plane.

Z(n) properties

$Z(n,T,V) = Z(-n,T,V)$ ⇒ $\xi \leftrightarrow \xi^{-1}$

$Z(n,T,V)$: **Real values** ⇒ $\xi \leftrightarrow \xi^*$

$\xi = e^{\mu_q/T} =$ ★



Phase Transition
N_{max} ~ infinity
(V ~ infinity)

Outline of strategy

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Integration method
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q} \ln Z_{\rm GC}$$

$$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$$

FT

Boyda's Talk

Experiments

Proton multiplicity data

$$P(n)$$

$$Z(n, T, V)$$

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

cut Baum-Kuchen algorithm

Lee-Yang zeros

Phase transition point

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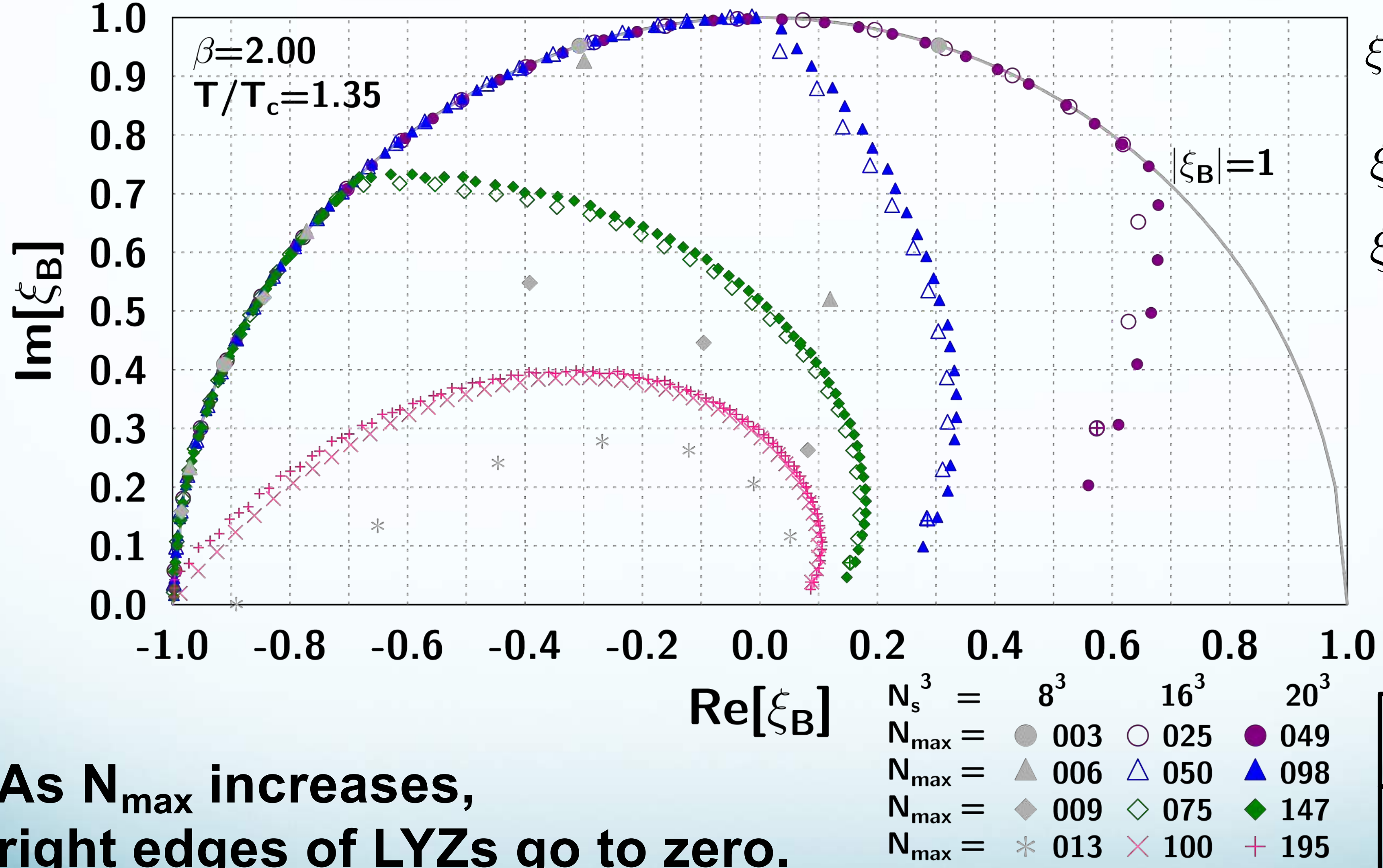
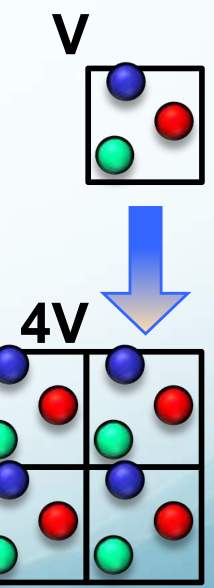
V and N_{max} dependences (T/T_c=1.35)



$\xi_B = e^{\mu_B/T}$

$\xi \leftrightarrow \xi^{-1}$

$\xi \leftrightarrow \xi^{*}$

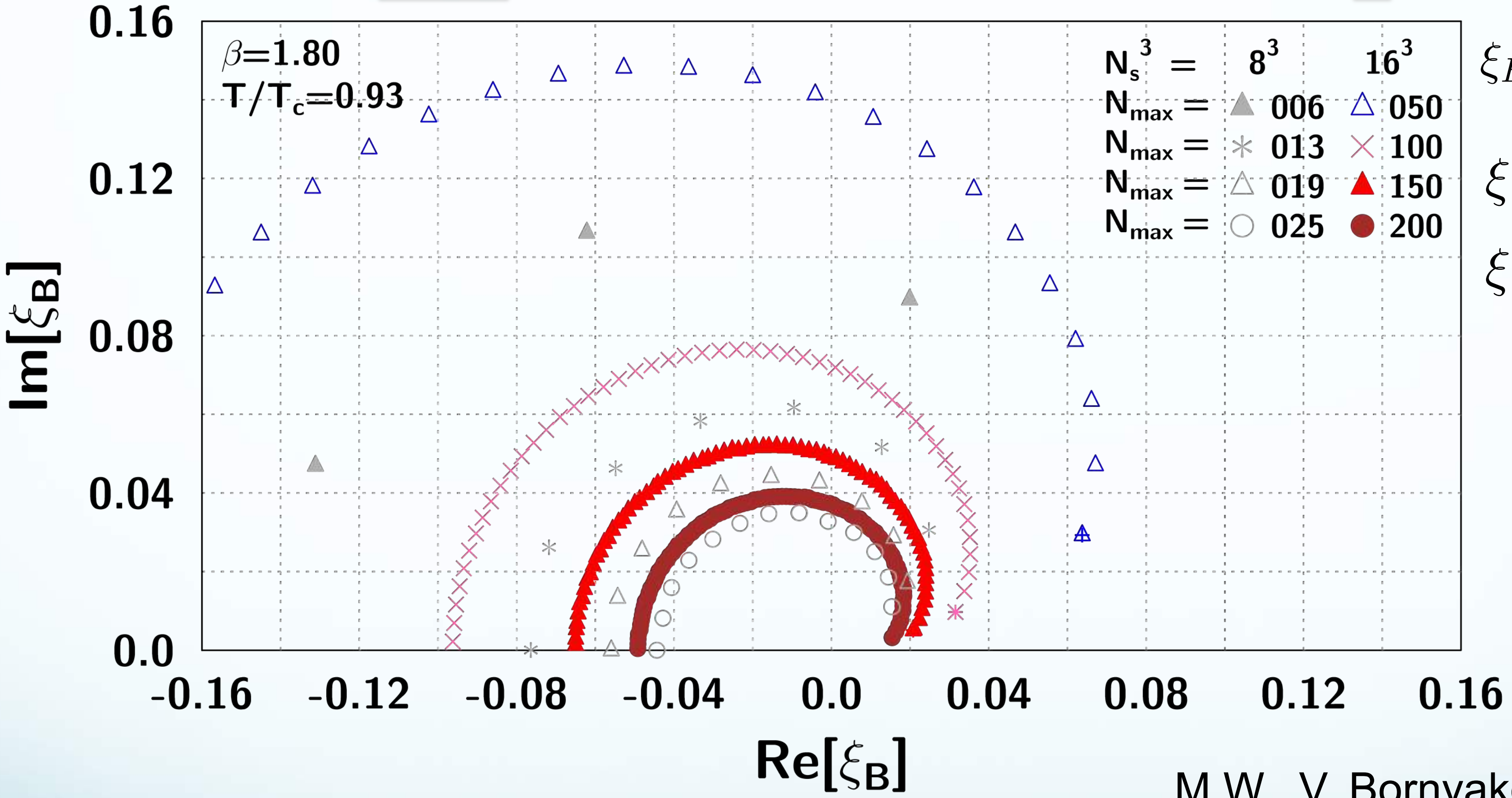


As N_{max} increases, right edges of LYZs go to zero.

$\xi_B = e^{\mu_B/T} = 0, \infty$: **No phase transition.**

arXiv:1802.02014[hep-lat]

V and N_{max} dependences (T/T_c=0.93)



$\xi_B = e^{\mu_B/T}$

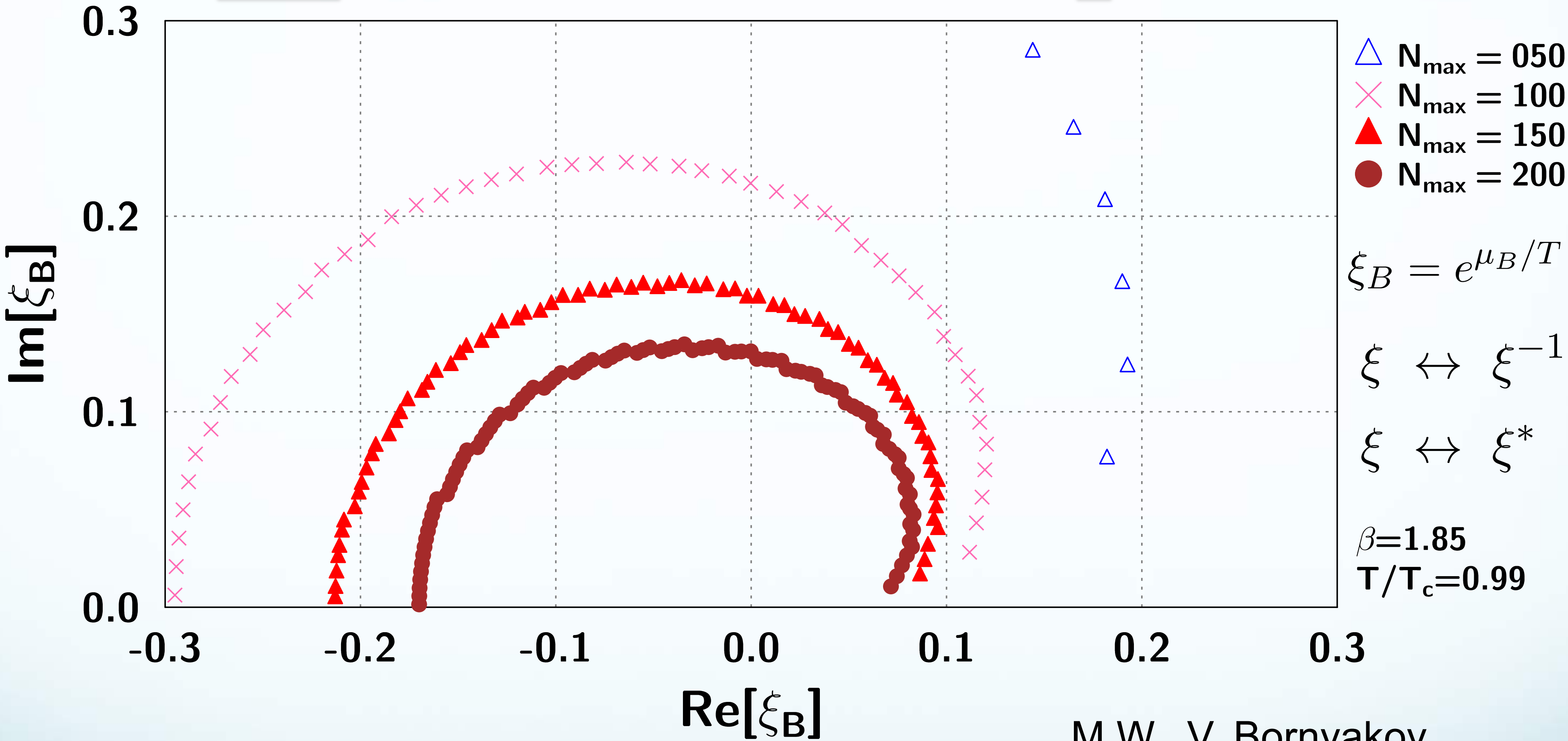
$\xi \leftrightarrow \xi^{-1}$

$\xi \leftrightarrow \xi^*$

As N_{max} increases, right edges of LYZs approach to the real positive axis.
Phase transition point: $\mu_B/T \sim$ 5-6 ?

M.W., V. Bornyakov, D. Boyda, V. Goy, H. Iida, A. Molochkov, A. Nakamura, V. Zakharov, arXiv:1802.02014[hep-lat]

N_{max} dependence (T/T_c=0.99)



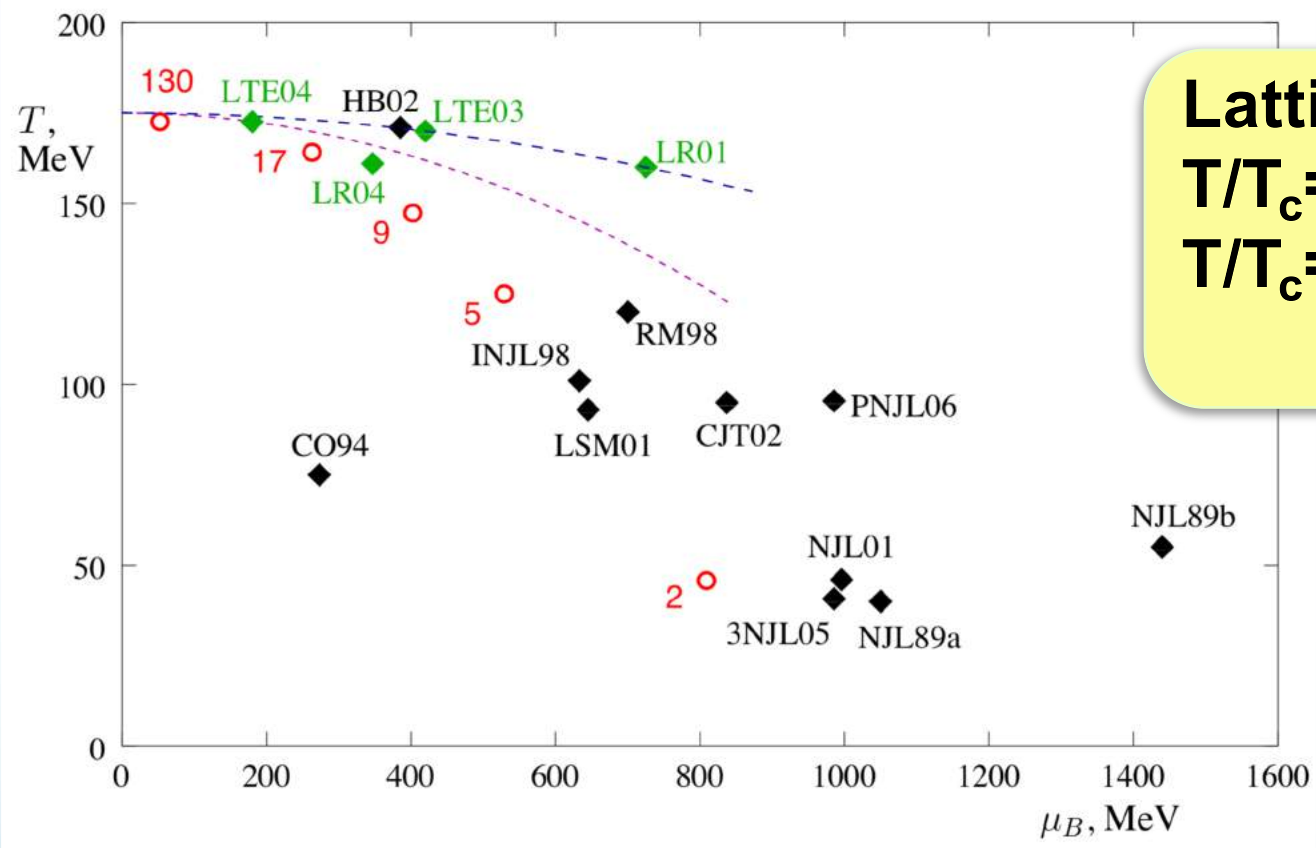
Phase transition point: $\mu_B/T \sim 3\text{-}3.5$ **?**

Is the extrapolation of LYZs work well?

M.W., V. Bornyakov,
D. Boyda, V. Goy,
H. Iida, A. Molochkov,
A. Nakamura, V. Zakharov,
arXiv:1802.02014[hep-lat]

QCD Phase diagram (Prediction)

M. Stephanov, PoS(Lattice2006), 024



Lattice Results

T/T$_c$=0.99: $\mu_B/T \sim 3\text{-}3.5$

T/T$_c$=0.93: $\mu_B/T \sim 5\text{-}6$

arXiv:1802.02014[hep-lat]

Integration method

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\sin}} f_k \sin(k\theta)$$

$$\theta = \frac{\mu_{qI}}{T}$$

We can roughly estimate the phase transition points from lattice QCD. But Is an extrapolation good? Are V and N$_{max}$ large enough? Is the number density approximation fine?

Outline of the strategy

NJL model

$n_q(\mu_q = i\mu_{qI}, T)$

Lattice QCD

$n_q(\mu_q = i\mu_{qI}, T, V)$

Integration method
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q} \ln Z_{\mathrm{GC}}$$

$Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$

Boyda's Talk

Experiments

FT

We can cross-check the extrapolation problem with the NJL model.

Proton multiplicity data

$Z(n, T, V)$ ← $P(n)$

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\mathrm{N_{max}}}^{\mathrm{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

NJL model

$n_q(\mu_q, T)$

cut Baum-Kuchen algorithm

Lee-Yang zeros

Phase transition point

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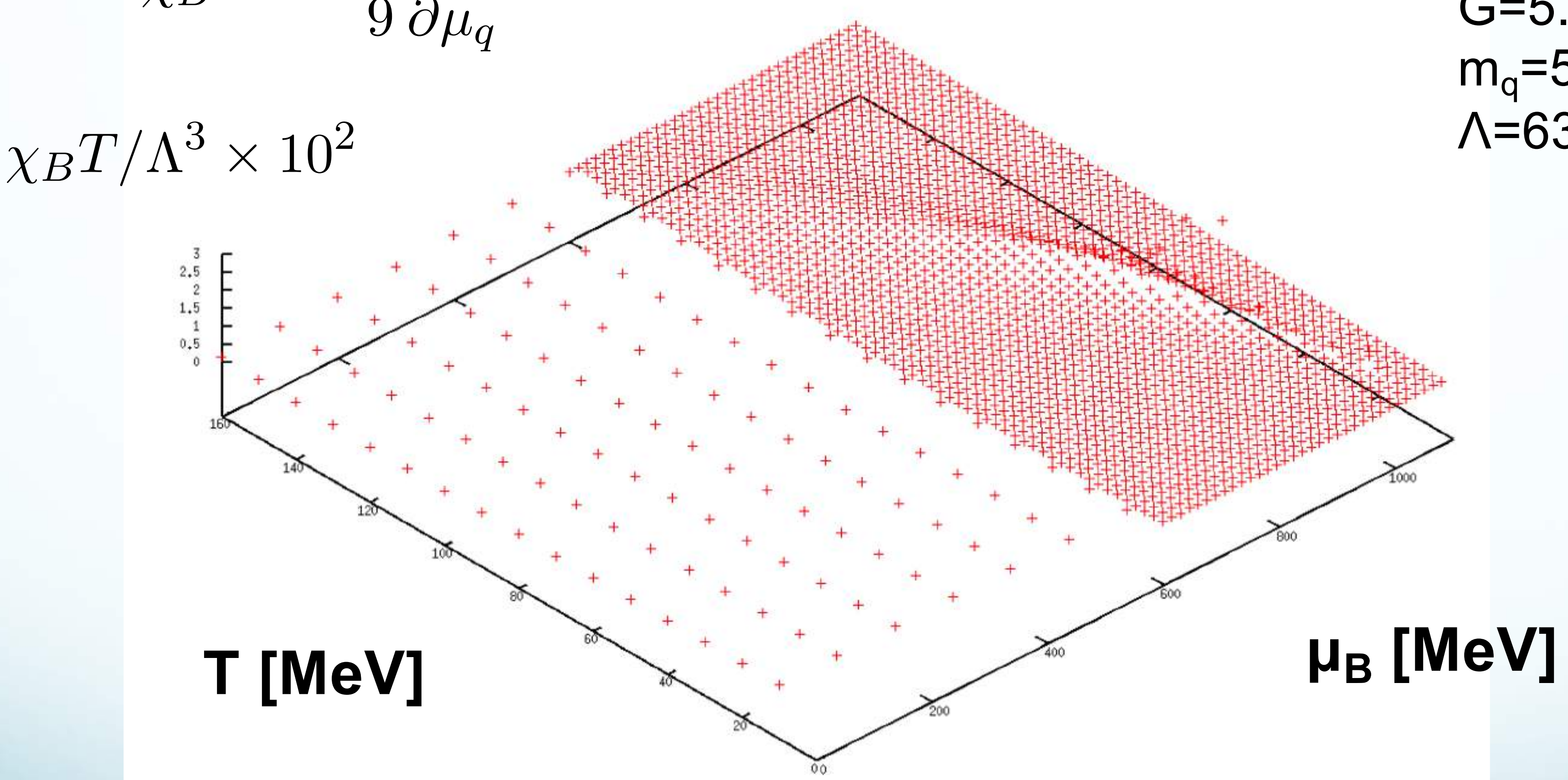
Susceptibility in the NJL model

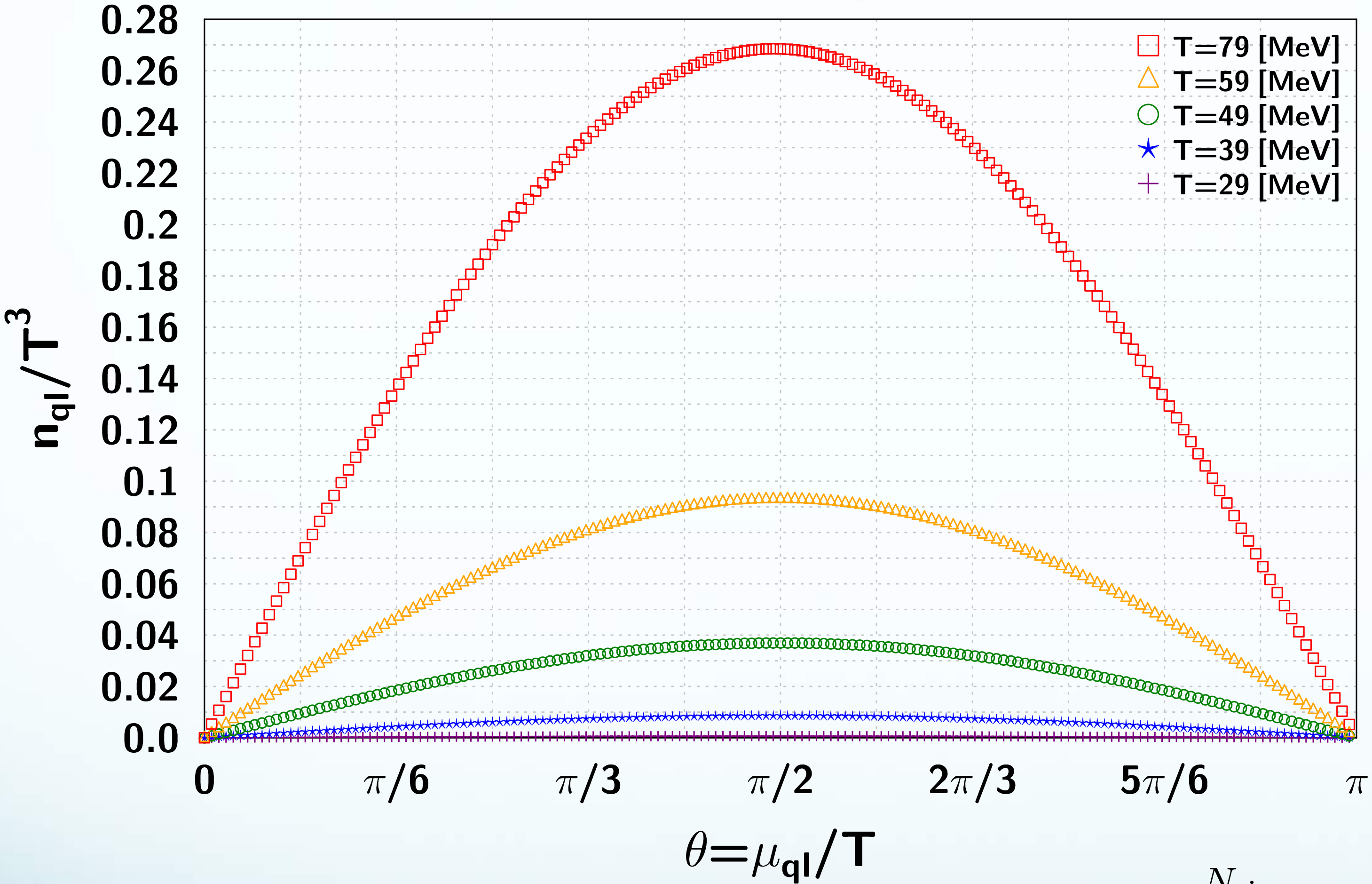
$$\chi_B = \frac{1}{9}\frac{\partial n_q}{\partial \mu_q}$$

Two-flavor NJL
$G=5.5\text{GeV}^{-2}$
$m_q=5.5\text{MeV}$
$\Lambda=631\text{MeV}$



Critical Point: $(T_{cp},\mu_B) \sim (49,981)$ [MeV]

Number density at pure imag. μ



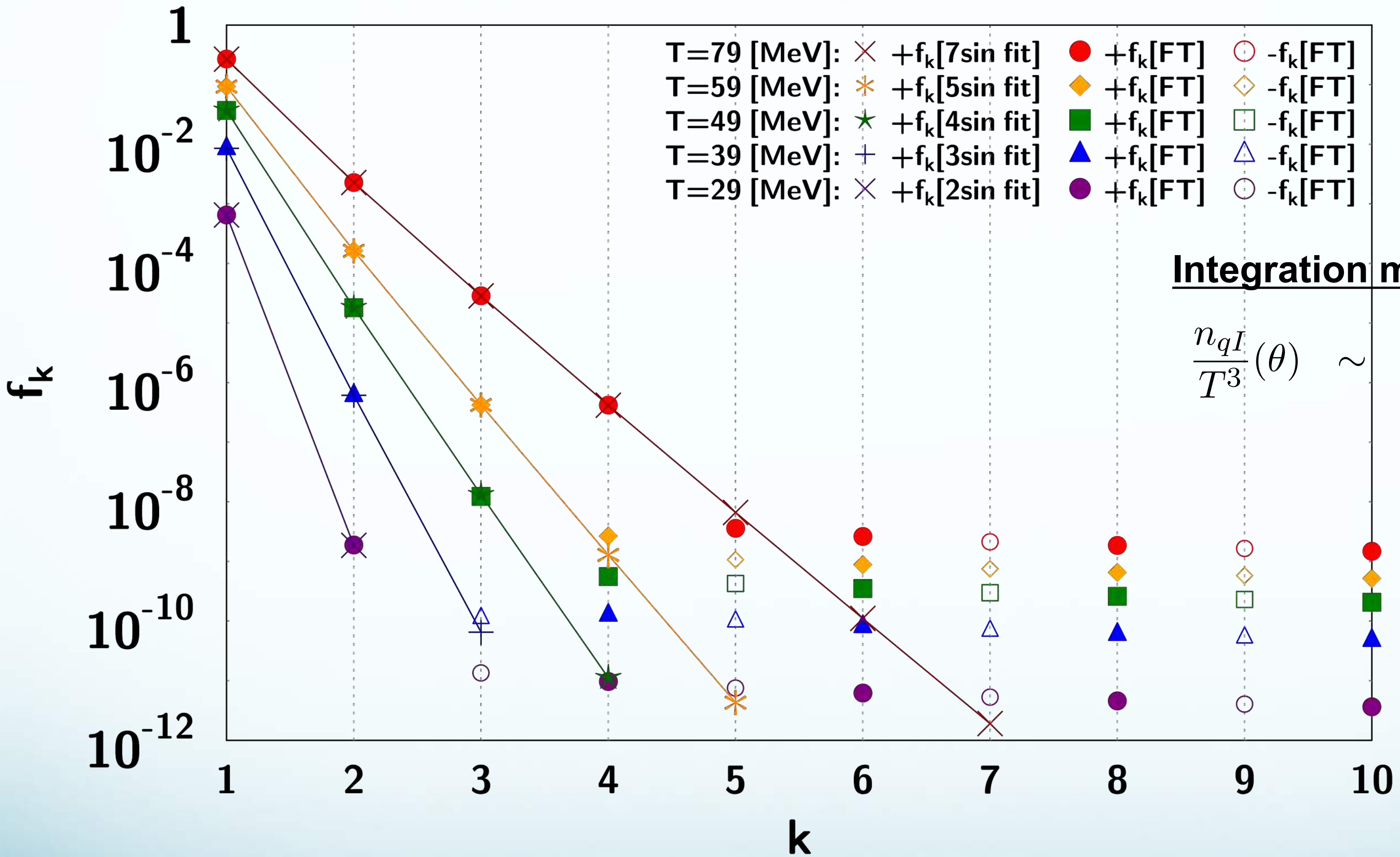
Number density

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$$n_q = i n_{qI}$$
$$\mu_q = i\mu_{qI}$$

We obtain coefficients f_k from $\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\sin}} f_k \sin(k\theta)$
as it was done in the lattice simulations.

Number density coefficients f_k



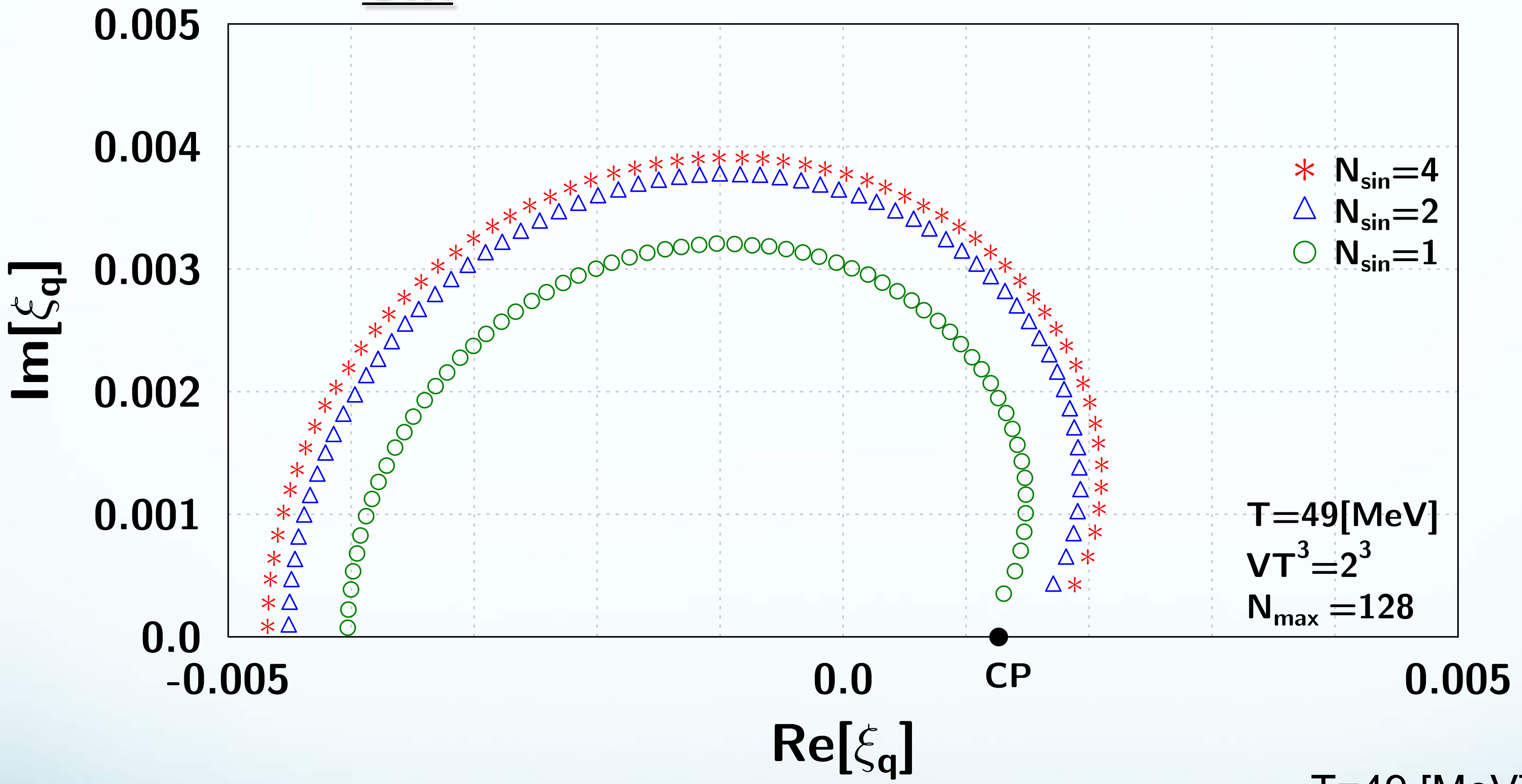
Integration method

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

$$\theta = \frac{\mu_{qI}}{T}$$

The values of f_k for small k are almost the same.

N_{sin} dependence of LYZs



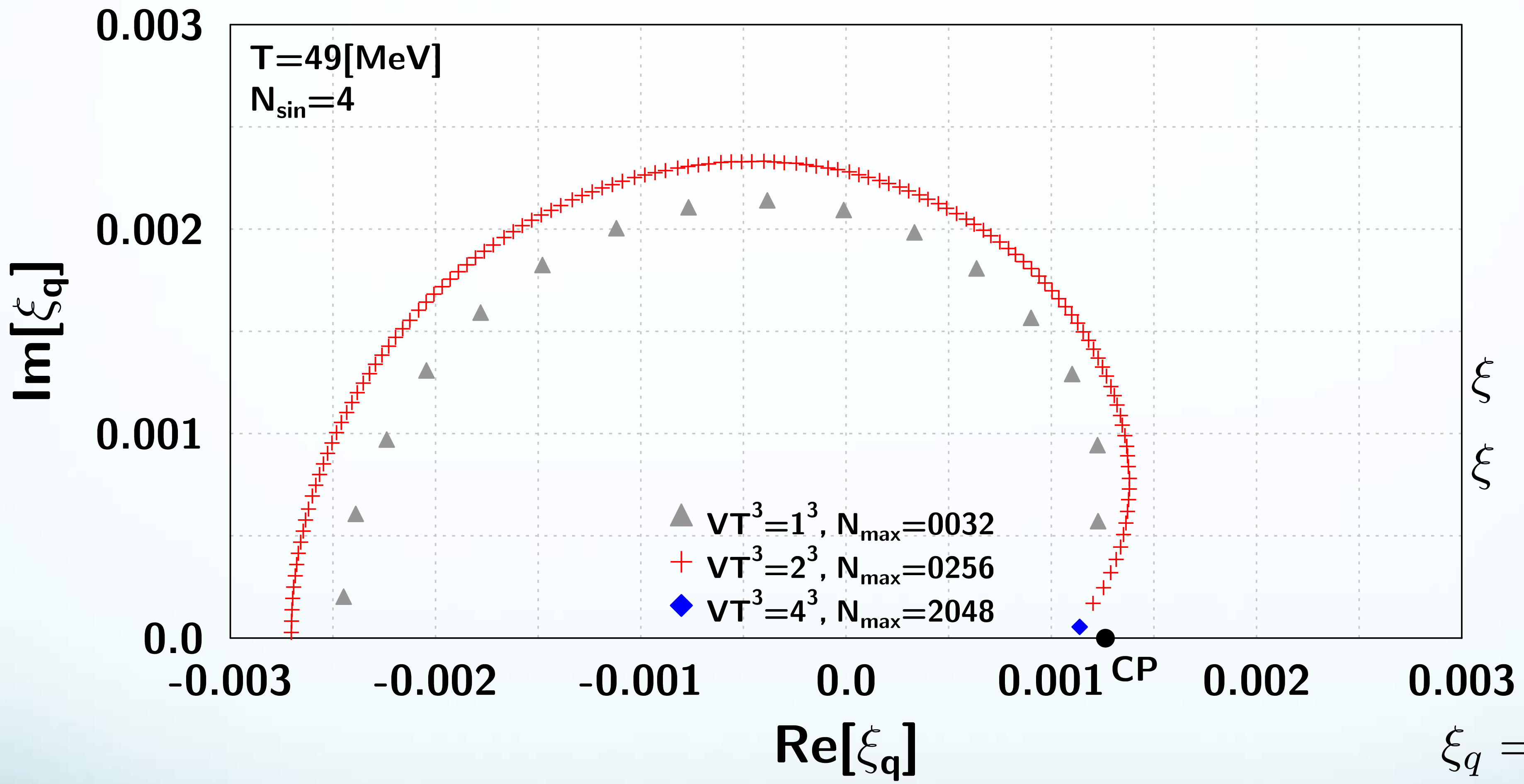
The LYZs for N_{sin}=2 is almost the same as one for N_{sin}=4.

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

T=49 [MeV]

$$f_2/f_1 = 4.9 \times 10^{-4}$$
$$f_3/f_1 = 3.6 \times 10^{-7}$$
$$f_4/f_1 = 3.0 \times 10^{-10}$$

V dependence of LYZs



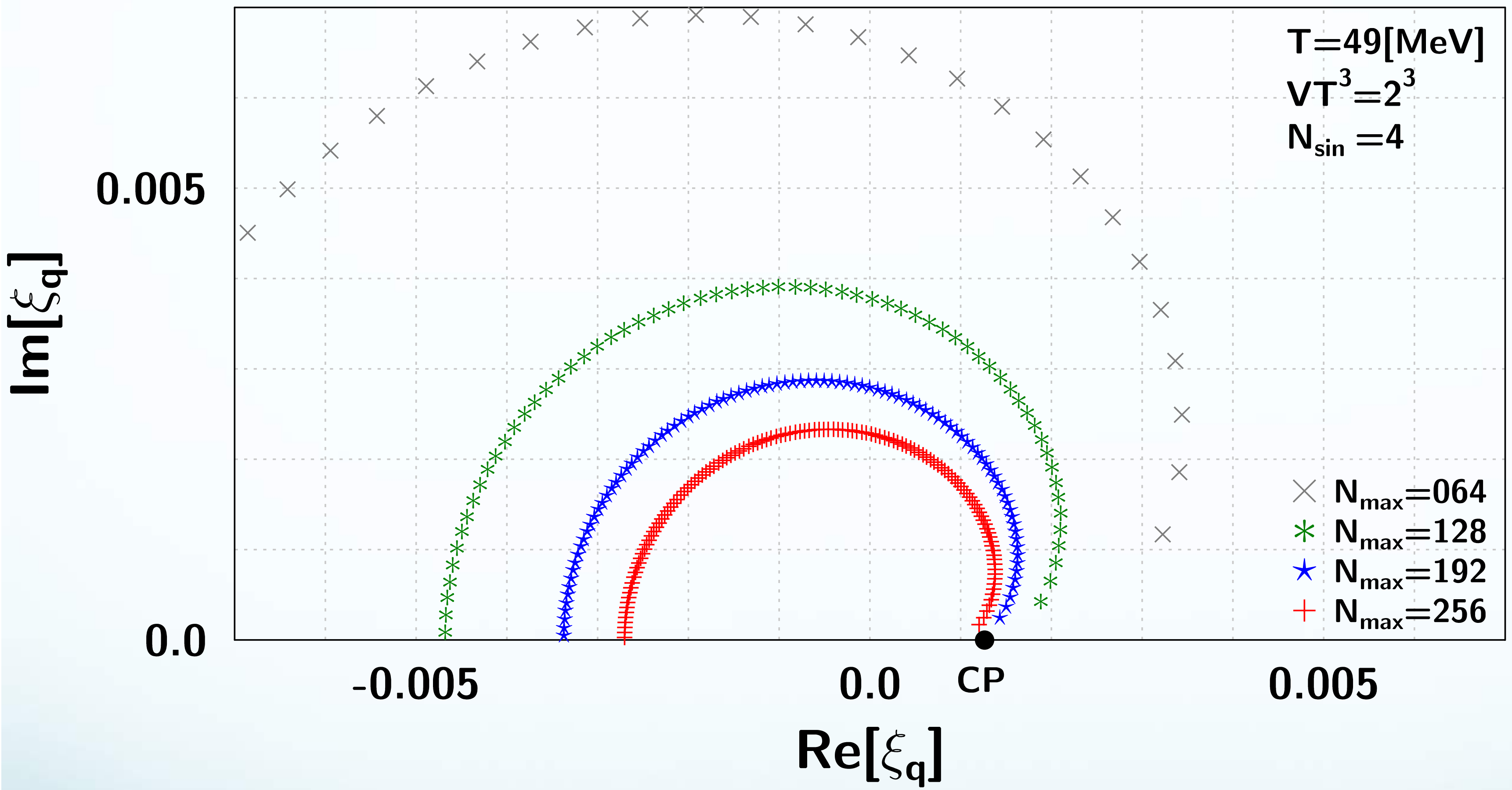
$\xi \leftrightarrow \xi^{-1}$

$\xi \leftrightarrow \xi^{*}$

$\xi_q = e^{\mu_q/T}$

As V increases for the fixed N_{max}/V, edges of LYZs approach to the real positive axis. The edge of LYZs for $VT^3=2^3$ is almost the same as one for $VT^3=4^3$.

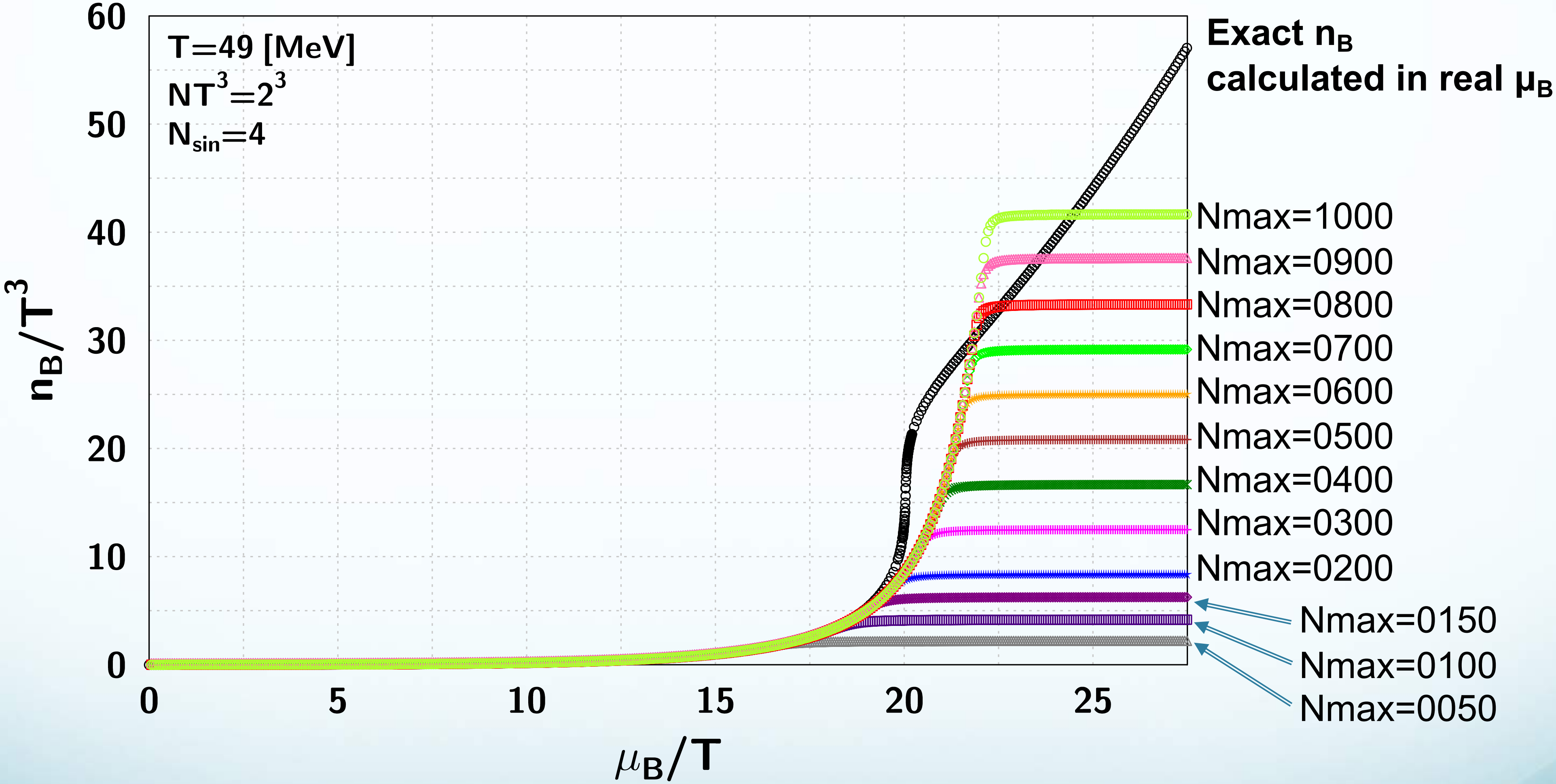
N_{max} dependence of LYZs at $T = T_{cp}$



$$\xi = e^{\mu_q/T}$$

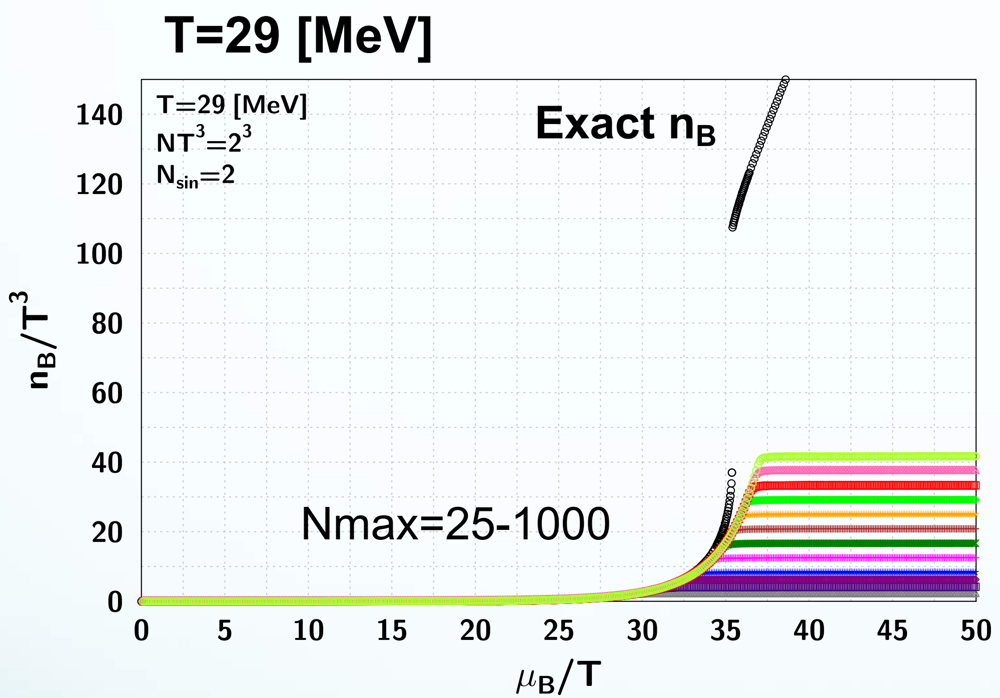
We can obtain the LYZs near the critical point (CP).

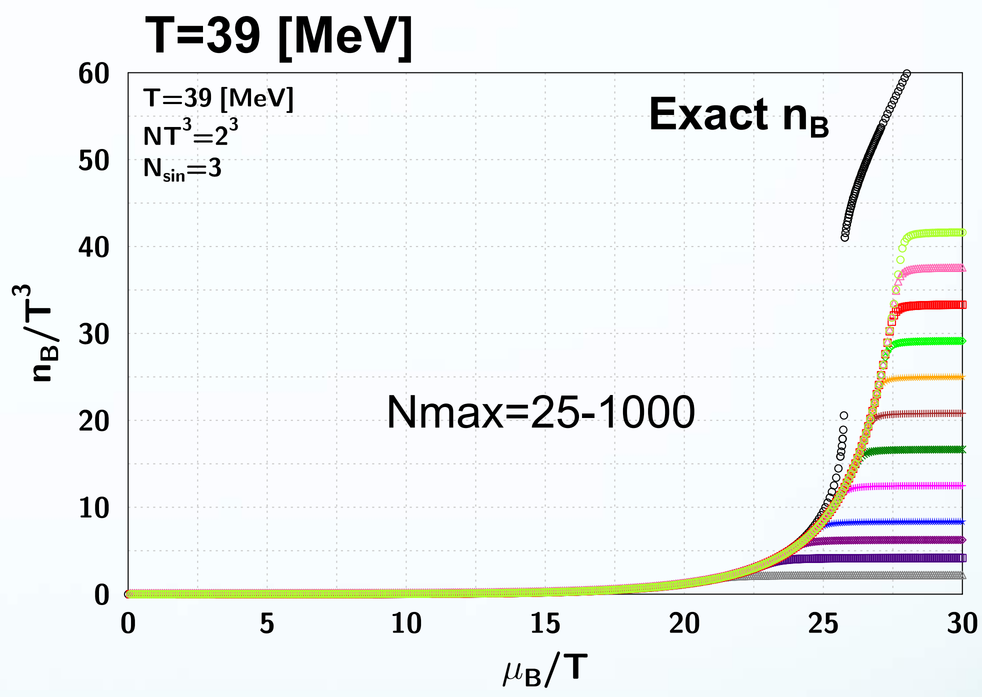
N_{max} dependence of n_B at $T = T_{cp}$



We can evaluate the exact n_B under the phase transition density from the canonical approach.

N_{max} dependence of n_B at $T < T_{cp}$

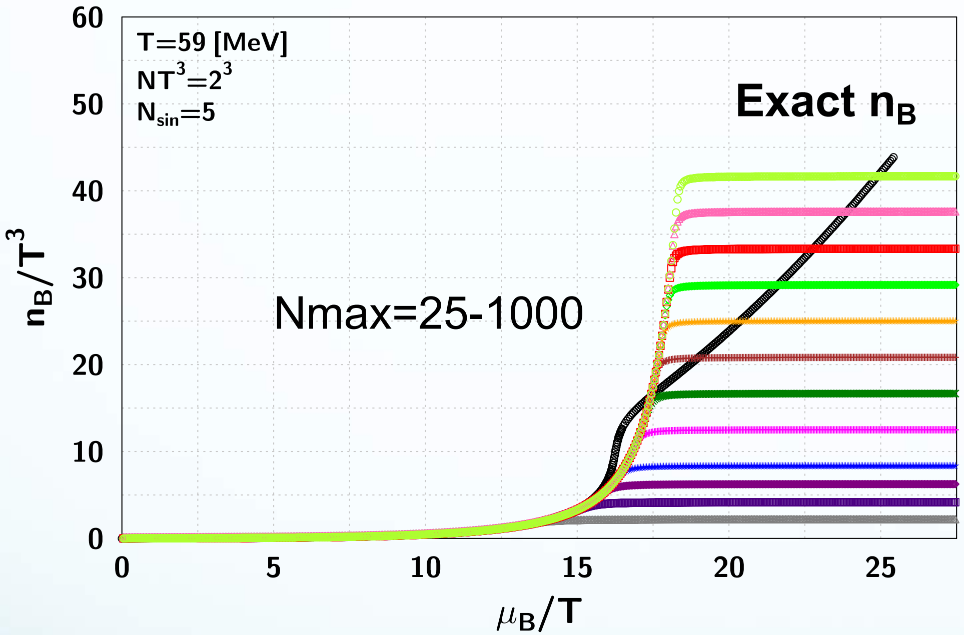




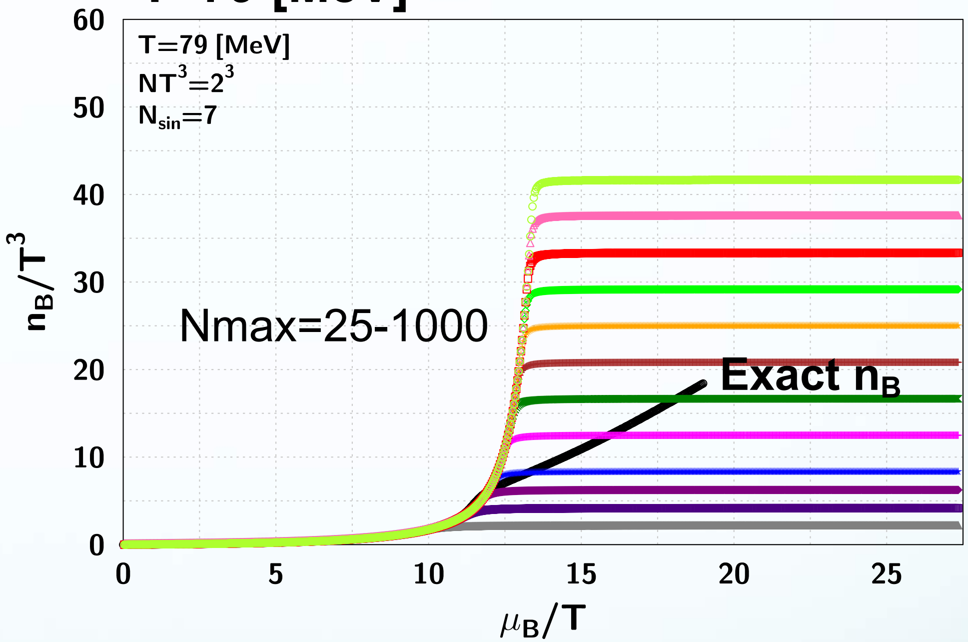
We can evaluate the exact n_B under the phase transition density from the canonical approach.

N_{max} dependence of n_B at $T > T_{cp}$

T=59 [MeV]

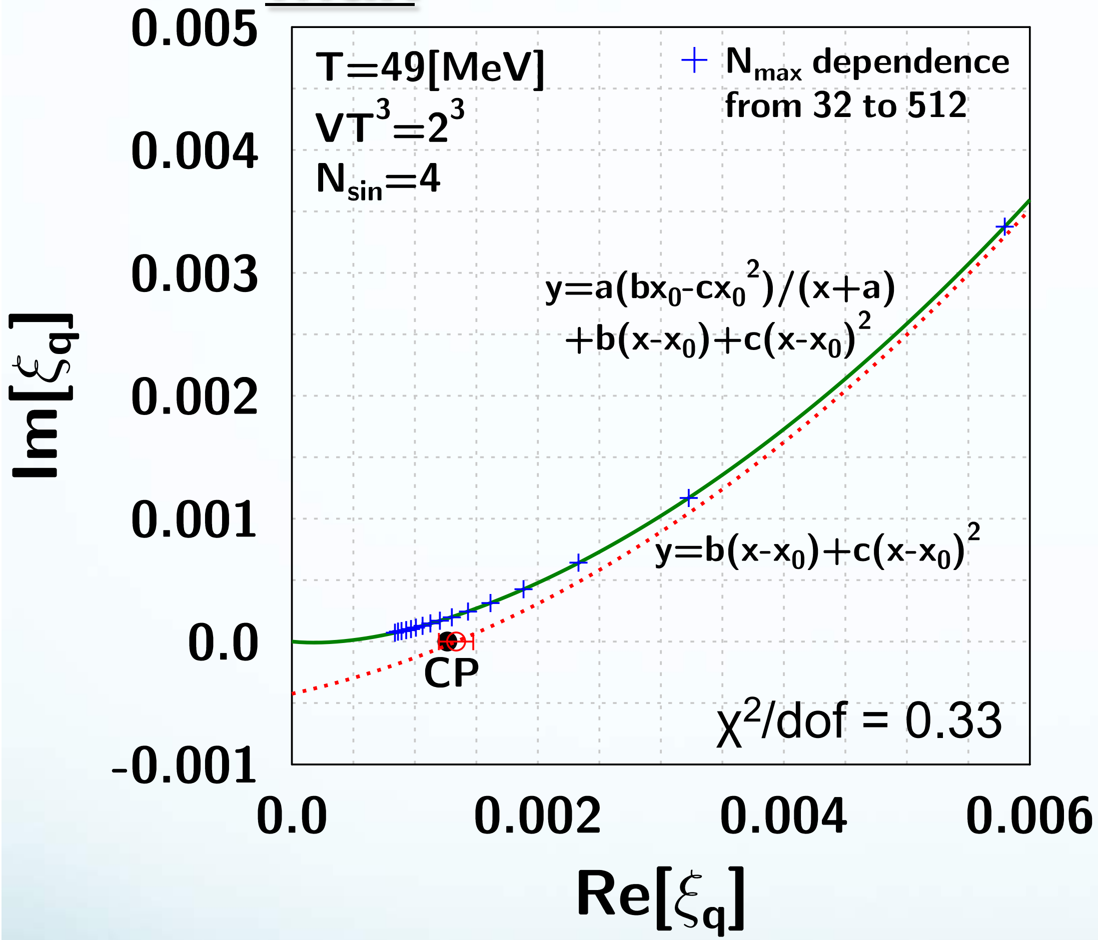


T=79 [MeV]



We can evaluate the exact n_B under the crossover density from the canonical approach.

N_{max} dependence of LYZs at T_{cp}



Number density approximation

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

Grand canonical partition function

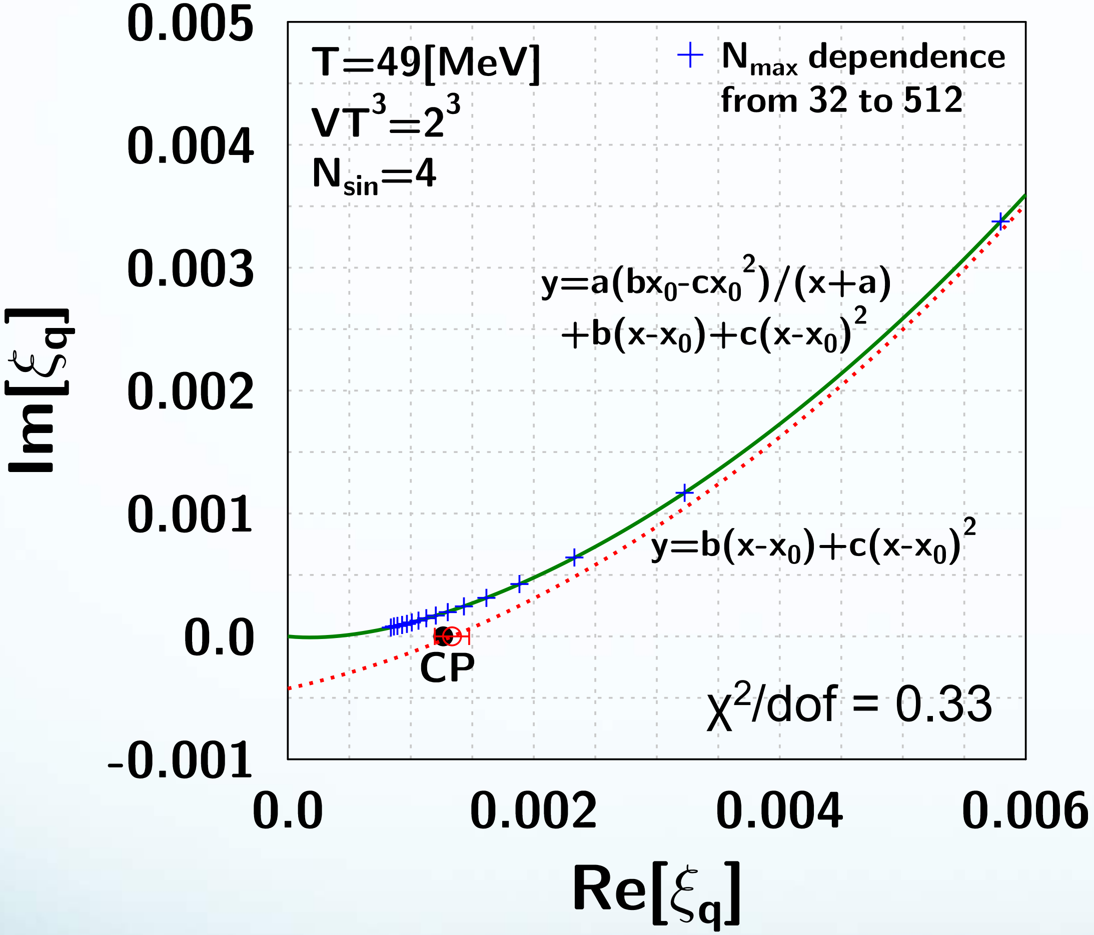
$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n$$

Number density

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial\mu_q}\ln Z_{\rm GC}$$

This extrapolation works well to obtain the phase transition points.

N_{max} dependence of LYZs at T = T_{cp}



$$\xi = e^{\mu_q/T}$$

Grand canonical partition function

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N}_{\max}}^{\mathbf{N}_{\max}} Z(n, T, V)\xi^n$$

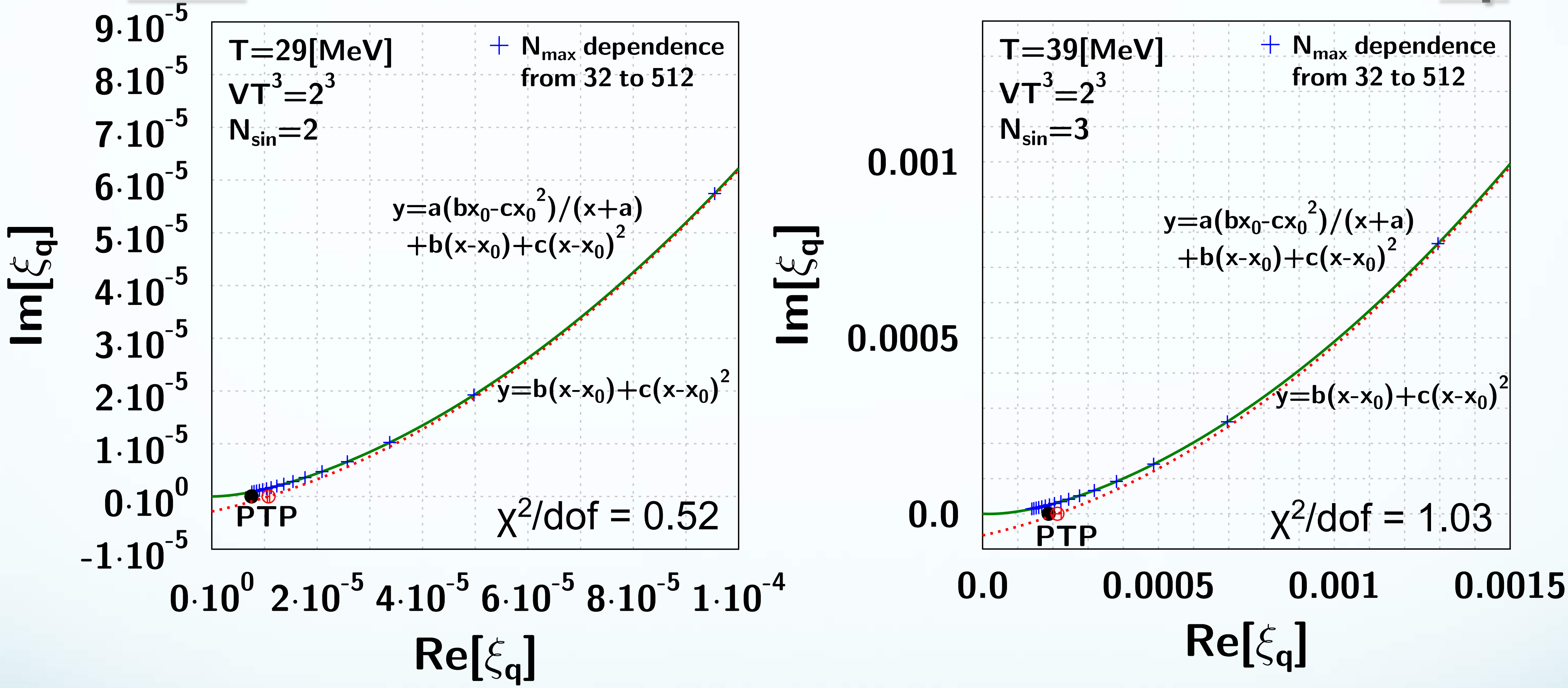
Number density

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

M.W., A. Hosaka, paper in preparation.

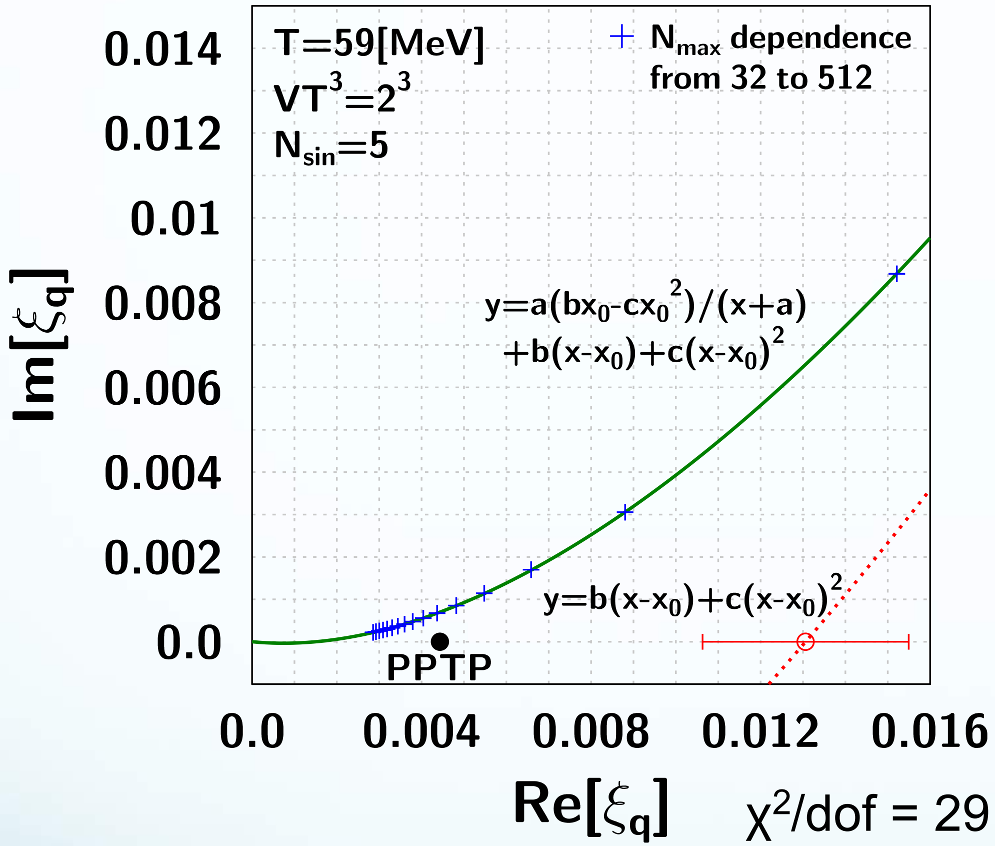
We have succeeded in subtracting a term associated with finite degree effect from the fitted function. The resulting curve represented the dotted curve nicely reproduces the expected critical point (CP) in the NJL model.

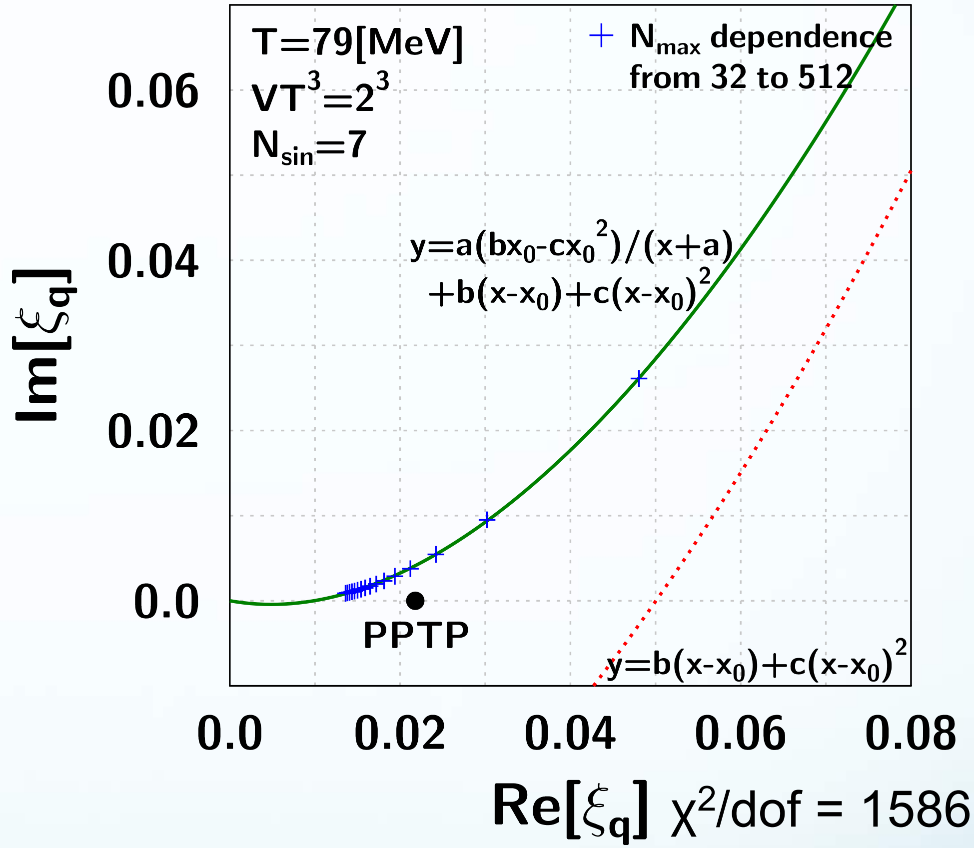
N_{max} dependence of LYZs at $T < T_{cp}$



This extrapolation procedure works well to obtain the expected phase transition points (PTP).

N_{max} dependence of LYZs at $T > T_{cp}$





Our results are different from the pseudo phase transition points (PPTP), which is consistent with lack of phase transition points at the real chemical potential.

Summary

- We studied Lee-Yang zeros for Z_n obtained from the canonical approach in lattice QCD and the NJL model.
- The phase transition points can be roughly estimated from lattice QCD.
- We checked V, N_{sin}, N_{max} dependences in the NJL model.
  - ・$VT^3 \geqq 2^3$ (L $\geqq$ 8 [fm])
  - ・$N_{sin} \geqq 2$ ($f_2/f_1 = 4.9 \times 10^{-4}$)
- We can evaluate the exact n_B under the phase transition density from the canonical approach.
- We found the reasonable extrapolation procedure of the edge of LYZs at $T \leqq T_{cp}$ in the NJL model.

Future work

- **Other Examples:**
  - **Polyakov-loop-extended NJL model**
    -- It has the Roberge-Weiss symmetry.
  - **SU(2)-color lattice**
    -- It does not have the sign problem.

- **Calculate SU(3)-color lattice with these parameters and determine the QCD phase!**