Canonical Approach: Analysis of RHIC data with LQCD

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HaRP workshop "Hadrons and dense matter from QCD",
Vladivostok, 27 - 29, January, 2019

Mission of this talk

This talk is about:

- Study of dense QCD with Canonical Approach (Lattice)

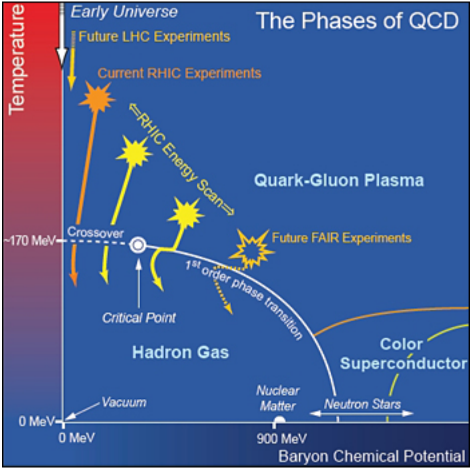
Main goals:

- Show results of dense QCD thermodynamics study
- Estimate our systematic: Compare with state of the art data
- Extract parameters of RHIC experiment, compare with previous statements

The outline

- Introduction: QCD Phase Diagram, Lattice QCD, Sign Problem
- Taylor expansion approach
- Analytical continuation approach
- Lattice Setup
- Number density for imagine chemical potential
- Comparisson with Taylor approach
- Canonical Approach: Integration method
- Thermodynamical observables
- Crossover line curvature estimation
- Comparison of our results with RHIC data
- Multiplicity: RHIC and Lattice
- Summary

QCD Phase Diagram



Experiments on heavy ion collisions:
RHIC, LHC, J-PARC, NICA

Theory
LQCD is only tool based on first principles calculation to study QCD Phase Diagram

but only for zero density (?)

Lattice QCD

Quantization with Lattice QCD

(C. Gattringer et.al. Quantum chromodynamics on the lattice (2010)):

$$\lim_{T\to\infty} \frac{1}{Z_T} tr\left[e^{-(T-t)\hat{H}}\hat{O}_2 e^{-t\hat{H}}\hat{O}_1\right] = \sum_n \langle 0|\hat{O}_2|n\rangle\langle n|\hat{O}_1|0\rangle e^{-tE_n}$$

$$\frac{1}{Z_T} tr\left[e^{-(T-t)\hat{H}}\hat{O}_2 e^{-t\hat{H}}\hat{O}_1\right] = \frac{1}{Z_T}\int \mathcal{D}\phi\, e^{-S_E}\, O_2[\phi]\, O_1[\phi]$$

- continuous space-time $\to$ discrete 4D Euclidean lattice with lattice spacing a
  - $S \to S_E$
  - $x \to a\,n, \quad n_\mu = 0, 1, 2, ...N-1$
  - $\psi(x) \to \psi(n)$, $A_\mu(x) \to U_\mu(n) = e^{iagA_\mu(n)}$
- Discretization of $S_E \to S^{lat}$ under $\lim\limits_{a\to 0} S^{lat} = S_E$
- Operators are translated to fuctions
- Euclidean correlates are computed on configurations generated with Boltzman probability $P(U) \propto e^{-S[U]}$

Lattice QCD

Standard Monte Carlo techniques

$$Z_{GC}(\mu, T, V) = \int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\; e^{-S_F - S_G} = \int \mathcal{D}U (\det \Delta(\mu))^{N_f} e^{-S_G}$$

Iwasaki Gauge Action

$$S_G = \frac{\beta}{3}\left(c_0 \sum_{x,\mu\nu} \mathrm{Re}\,\mathrm{Tr}\,\left(1 - U^{1\times 1}_{\mu\nu}(x)\right) + c_1 \sum_{x,\mu\nu} \mathrm{Re}\,\mathrm{Tr}\,\left(1 - U^{1\times 2}_{\mu\nu}\right)\right),$$

Fermion Action (Improved Wilson Action) $S_f = \sum_{f=u,d} \sum_{x,y} \bar{\psi}^f_x \Delta_{x,y} \psi^f_y$

$$\begin{aligned}
\Delta_{x,y}(\mu) \;=\;& \delta_{xy} - \kappa \sum_{i=1}^{3} \{(1-\gamma_i) U_{x,i} \delta_{x+\hat{i},y} + (1+\gamma_i) U^{\dagger}_{y,i} \delta_{x,y+\hat{i}}\} \\
& - \kappa \{ e^{a\mu_q} (1-\gamma_4) U_{x,4} \delta_{x+\hat{4},y} + e^{-a\mu_q} (1+\gamma_4) U^{\dagger}_{y,4} \delta_{x,y+\hat{4}} \} \\
& - \delta_{xy} c_{SW} \kappa \sum_{\mu<\nu} \sigma_{\mu\nu} P_{\mu\nu}
\end{aligned}$$

Sign problem

Standard Monte Carlo techniques

$$Z_{GC}(\mu, T, V) = \int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\; e^{-S_F - S_G} = \int \mathcal{D}U (\det\Delta(\mu))^{N_f} e^{-S_G}$$

Monte Carlo works: Boltzman weight $P(U) \propto e^{-S[U]}$ is positive

- μ is real $\to$ $\det\Delta(\mu)$ is complex
- μ is imaginary $\to$ $\det\Delta(\mu)$ is real
  (determinant satisfies relation $[\det\Delta(\mu)]^* = \det\Delta(-\mu^*)$)

Approaches

- zero density (μ is real) $\to$ Taylor Expansion method
- imagine density (μ is imaginary) $\to$ Analytical Continuation

Taylor expansion approach - existing data

Expand Logarithm of Partition Fuction in power of μ:

$$\frac{P}{T^4} = \frac{1}{VT^3} logZ_{GC}(\mu) = \sum_{n=0}^{\infty} \chi_n(T) \left(\frac{\mu}{T}\right)^n, \quad \chi_n(T) = \frac{1}{n!} \frac{N_t^3}{N_s^3} \left. \frac{\partial^n ln\, Z_{GC}}{\partial(\mu/T)^n} \right|_{\mu=0}$$

And calculate χ_n using Monte Carlo at $\mu = 0$:

$$\chi_n \propto \langle \mathrm{Tr}\, \Delta^{-1} \Delta' \, \Delta^{-1} \Delta' \ldots \Delta^{-1} \Delta' \rangle$$

- state of the art
  The QCD Equation of Stati to $\mathcal{O}(\mu_B^6)$ from Lattice QCD//
  A. Bazarov et. al. Phys. Rev. D 95, 054504 (2017)
  $T/T_c \in [0.84, 2.14]$, $\mu_B/T <= 2,\ T_c = 154 \pm 9 Mev$
- same lattice action
  Equation of State and Heavy-Quark Free Energy at Finite Temperature and Density in Two Flavor Lattice QCD with Wilson Quark Action
  S. Ejiri et. al. Phys. Rev. D 82, 014508 (2010)
  $T/T_c \in [0.84, 2.)\ \mu_B/T <= 3.6$

Analytical continuation approach - comparisson

Using Monte Carlo at imaginary μ calculate number density:

$$n_B^{lat} = \frac{1}{3N_sN_t}\frac{\partial \ln Z_{GC}}{\partial(a\mu)} = \frac{N_f}{3N_s^3N_t}\int \mathcal{D}Ue^{-S_G}\mathrm{tr}\left[\Delta^{-1}\frac{\partial\Delta}{\partial(a\mu)}\right](det\Delta(\mu))^{N_f}$$

And fit it as function of μ:

- Deconfinement: $f^{pol}(x) = \sum_k^{k_{max}} a_k\ (\mu/T)^{2k+1}$
- Confinement: $f^{sin}(x) = \sum_k^{k_{max}} f_k\, sin(k\,\mu/T)$

J. Takahashi et.al. Phys. Rev. D 91, 014501 (2014)

T/T_{c0}	$a_F^{(1)}$	$a_F^{(2)}$	$a_F^{(3)}$	χ^2/d.o.f.	μ_I/T(fitting range)
0.93	0.250(2)			5.937	$0 \sim \pi/3$
0.93	0.251(2)	−0.00457(216)		6.084	$0 \sim \pi/3$
0.93	0.251(2)	−0.00526(219)	0.00440(214)	6.290	$0 \sim \pi/3$
0.99	0.718(2)			11.06	$0 \sim \pi/3$
0.99	0.728(3)	−0.0179(26)		7.453	$0 \sim \pi/3$
0.99	0.727(3)	−0.0137(30)	−0.00825(276)	7.288	$0 \sim \pi/3$

Ours

T/T_c	f_3	f_6	a_1	a_3	a_5	χ^2/N_{dof}	N_{dof}	$2c_2$
0.99	0.7326(25)	-0.0159(21)	2.102(5)	-2.719(17)	0.453(55)	0.83	18	2.071(34)
0.93	0.2608(8)	-	0.7824(24)	-1.1736(36)	0.5281(16)	0.93	37	0.713(40)
0.84	0.0844(7)	-	0.2532(21)	-0.3798(31)	0.1709(14)	0.41	18	0.251(35)

Our Lattice Setup

- clover improved Wilson action
- Iwasaki gauge action
- Lattice $4x16^3$ ($L \approx 3.2fm$, $a \approx 0.2fm$)
- $\frac{m_\pi}{m_\rho} = 0.8 \quad (m_\pi = 0.7GeV)$
- $T/T_c = 0.84, 0.93, 0.99, 1.08, 1.20, 1.35$
- 40 values μ_I, 1800 - 3800 configurations (10 trajectories separated)

Parameters were taken from
S. Ejiri et. al., PRD 82, 014508 (2010)

Our cluster: Vostok1 (20 GPU K40)

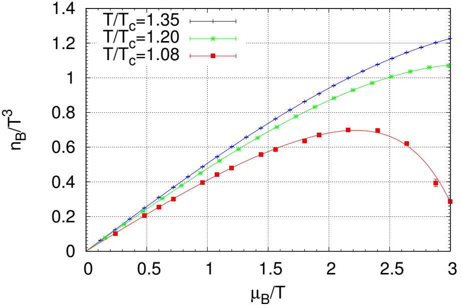
Number density $n_B^{lat}(\mu_i)$ at **imagine** μ_I: Lattice data and Fitting

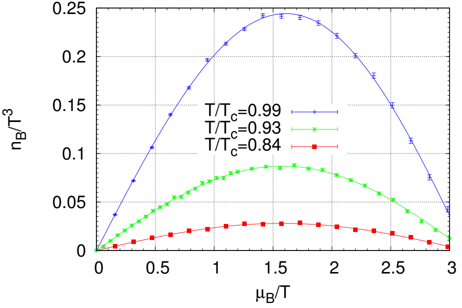
Taylor Series $(T > T_c)$

$$f^{pol}(x) = \sum_k^{k_{max}} a_k\, x^{2k+1}$$

Fourier Series $(T < T_c)$

$$f^{sin}(x) = \sum_k^{k_{max}} f_k\, sin(k\,x), \quad x = \mu_I/T$$

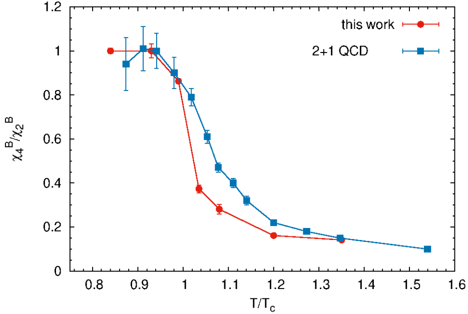


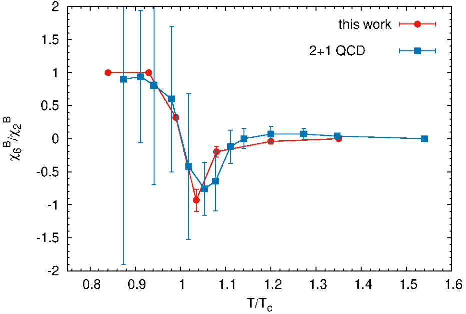


- Presicions increased
- More series terms extracted

Comparisson with Taylor expansion

$$\frac{P}{T^4} = \frac{1}{VT^3} logZ_{GC}(\mu) = \sum_{n=0}^{\infty} \chi_n(T) \left(\frac{\mu}{T}\right)^n, \quad \chi_n(T) = \frac{1}{n!} \frac{N_t^3}{N_s^3} \left.\frac{\partial^n ln Z_{GC}}{\partial(\mu/T)^n}\right|_{\mu=0}$$





2+1 QCD - The QCD Equation of Stati to $\mathcal{O}(\mu_B^6)$ from Lattice QCD // A. Bazarov et. al. Phys. Rev. D 95, 054504 (2017)

Surprising agreement with state of the art data (physical quark mass)!

Canonical approach

By definition

$$Z_{GC}(\mu, T, V) = Tr(e^{-\frac{\hat{H}-\mu\hat{N}}{T}}) = \sum_{n=-\infty}^{\infty} < n|e^{-\frac{\hat{H}}{T}}|n > e^{\frac{\mu n}{T}} =$$

$$= \sum_{n=-\infty}^{\infty} Z_C(n, T, V) e^{\frac{\mu n}{T}} = \sum_{n=-\infty}^{\infty} Z_n \xi^n$$

$Z_C(n, T, V)$ - canonical partition function (Z_n in following)
$\xi = e^{\mu/T}$ - fugacity
$\hat{N}$ - operator of any conserved quantum number (baryon, charge, etc.)

For imaginary μ we can calculate Z_n by inverse Fourier transformation (A. Hasenfratz and D. Toussaint, Nucl. Phys. B 371 (1992))

$$Z_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} Z_{GC}(\mu = i\theta T, T, V).$$

Integration method

Idea: *Using lattice data of baryon density $n_B(\mu)$ calculation, restore Grand Canonical Partition Function $Z_{GC}(\mu/T)$*

For imagine chemical potential ($\mu = i\mu_I$, $\theta = \mu_I/T$):

$$n_B = \frac{1}{V}\frac{\partial(lnZ_{GC})}{\partial(\mu/T)} \quad \rightarrow \quad lnZ_{GC}(\theta) - lnZ_{GC}(0) = V\int_0^{\theta} d(i\tilde{\theta})\, i\, Im[n_B(\tilde{\theta})]$$

$$\Rightarrow \frac{Z_{GC}(\theta)}{Z_{GC}(0)} = exp\left(-V\int_0^{\theta} dx\, Im[n_B(x)]\right)$$

$$Z_n/Z_0 = \frac{\int_0^{2\pi}\frac{d\theta}{2\pi} e^{-in\theta}\frac{Z_{GC}(\theta)}{Z_{GC}(0)}}{\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{Z_{GC}(\theta)}{Z_{GC}(0)}} = \frac{\int_0^{2\pi}\frac{d\theta}{2\pi} e^{-in\theta} e^{-V\int_0^{\theta} dx\, Im[n_B(x)]}}{\int_0^{2\pi}\frac{d\theta}{2\pi} e^{-V\int_0^{\theta} dx\, Im[n_B(x)]}},$$

where $Im[n_B(\theta)]$ - Monte Carlo data

- numerical integration
- use of some parametrization ($f^{pol}(x)$ or $f^{sin}(x)$) for $n_B(x)$

Integration method STRATEGY

1. Calculate $n_B(\mu_I/T)$ using LQCD simulation at imaginary μ_I
$$n_B^{lat} = \frac{1}{3N_sN_t}\frac{\partial \ln Z_{GC}}{\partial(a\mu)} = \frac{N_f}{3N_s^3N_t}\int \mathcal{D}Ue^{-S_G}\mathrm{tr}\left[\Delta^{-1}\frac{\partial\Delta}{\partial(a\mu)}\right](det\Delta(\mu_I))^{N_f}$$

2. Fit lattice data of number density to some function $n_B^{lat} \approx f(\mu_I/T)$
   Deconfinement: $f^{pol}(x) = \sum_k^{k_{max}} a_k\,(\mu_I/T)^{2k+1}$
   Confinement: $f^{sin}(x) = \sum_k^{k_{max}} f_k\, sin(k\,\mu_I/T)$

3. Restore $Z_{GC}(\mu_I)$
$$\frac{Z_{GC}(\theta)}{Z_{GC}(0)} = exp\left(-V\int_0^\theta d(\tfrac{\mu_I}{T})\, f(\mu_I/T)\right)$$

4. Calculate Z_n as Fourier transformation of $Z_{GC}(\mu_I)$
$$Z_n/Z_0 = \frac{\int_0^{2\pi}\frac{d\theta}{2\pi}e^{-in\theta}Z_{GC}(\theta)/Z_{GC}(0)}{\int_0^{2\pi}\frac{d\theta}{2\pi}Z_{GC}(\theta)/Z_{GC}(0)}$$

5. Using Z_n calculate $Z_{GC}(\mu)$ at **real** μ
$$Z_{GC}(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{\frac{\mu n}{T}}$$
   and thermodynamic observables $\left(n_B = \frac{1}{V}\frac{\partial(lnZ_{GC})}{\partial(\mu/T)} = C\,\frac{2\sum_n^{n_{max}} nZ_n \sinh(n\mu/T)}{1+2\sum_n^{n_{max}} Z_n\cosh(n\mu/T)}\right)$

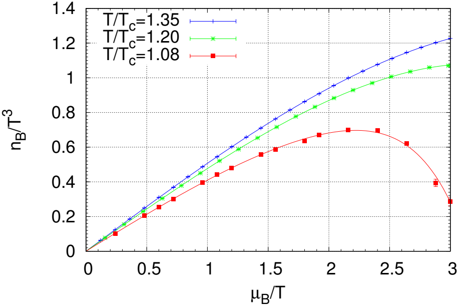
Number density $n_B^{lat}(\mu_i)$ at imagine μ_I: Lattice data and Fitting

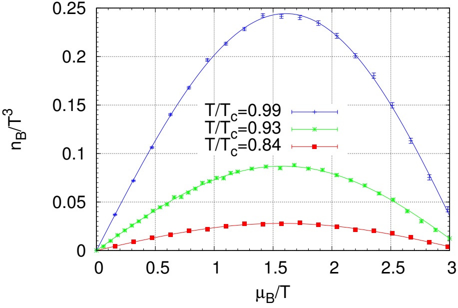
Taylor Series $(T > T_c)$

$$f^{pol}(x) = \sum_k^{k_{max}} a_k x^{2k+1}$$

Fourier Series $(T < T_c)$

$$f^{sin}(x) = \sum_k^{k_{max}} f_k \sin(k\,x), \quad x = \mu_I/T$$

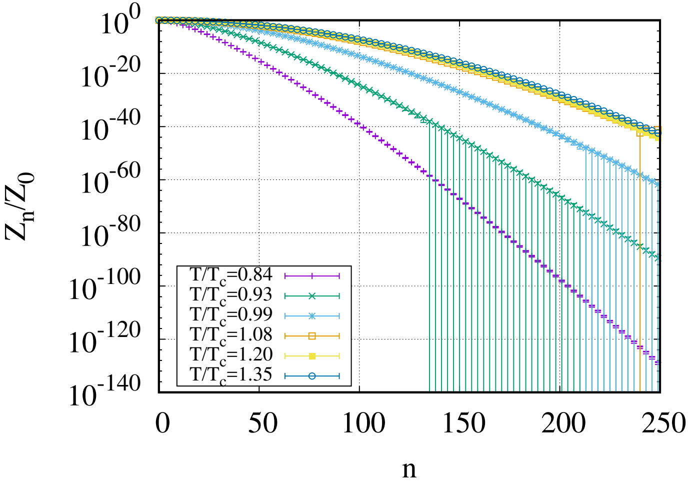




- Presicions increased
- More series terms extracted

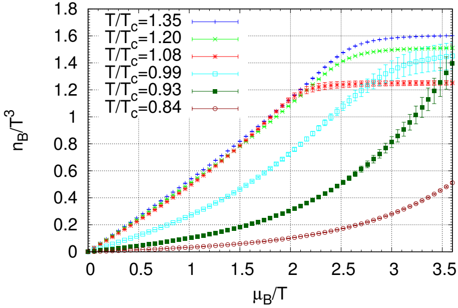
Canonical partition functions Z_n

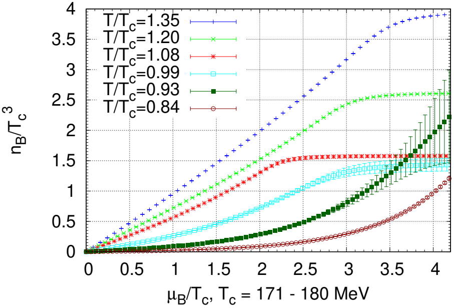
$$Z_n/Z_0 = \frac{\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} e^{L_Z(\theta)}}{\int_0^{2\pi} \frac{d\theta}{2\pi} e^{L_Z(\theta)}}, \quad L_Z(\theta) = -V \int_0^{\theta} dx\, Im[n_B(x)]$$



Baryon density

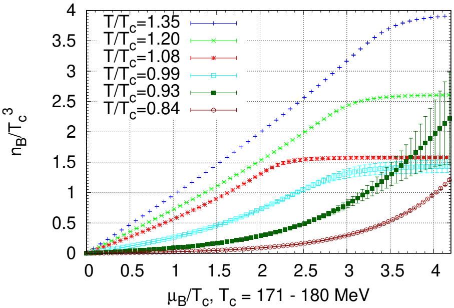
$$Z_{GC}(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{\frac{\mu n}{T}} \quad \Rightarrow n_B = \frac{1}{V}\frac{\partial(lnZ_{GC})}{\partial(\mu/T)} = C\frac{2\sum_n^{n_{max}} nZ_n \sinh(n\mu/T)}{1+2\sum_n^{n_{max}} Z_n \cosh(n\mu/T)}$$

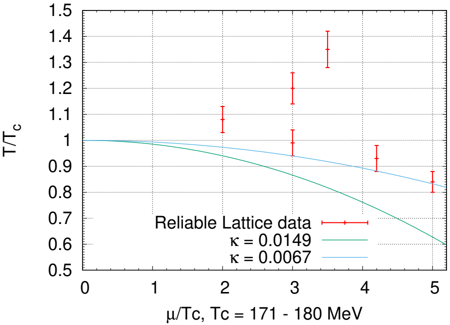




Crossover transition line curvature estimation

$$T_c(\mu_B) = T_c(0)\left(1 - \kappa\left(\frac{\mu_B}{T_c(0)}\right)^2\right)$$





- $\kappa = 0.0149(21)$ - The QCD phase diagram from analytic continuation // R. Bellwied et. al. (2015) arXive:1507.07510
- $\kappa = 0.0066(7)$ - The QCD phase diagram at nonzero quark density // G.Endrodi et. al. JHEP 1104:001, 2011

Comparisson with RHIC experiment

A. Nakamura, K. Nagata PTEP, 033D01 (2016)
RHIC STAR data **(Luo X. CEJP 10, 1372 (2012))**

Probability interpretation:

$$1 = \sum_n \frac{Z_n \xi^n}{Z_{GC}(\mu)}, \quad \xi = e^{\mu/T} \qquad \frac{N(n)}{N(0)} = \frac{N\, Z_n \xi^n / Z_{GC}(\mu)}{N\, Z_0 \xi^0 / Z_{GC}(\mu)} = Z_n \xi^n / Z_0$$

Multiplicity: $P_n = Z_n \xi^n \quad \Rightarrow \quad Z_n = P_n P_{-n}$ и $\xi = \sqrt[2n]{\frac{P_n}{P_{-n}}}$

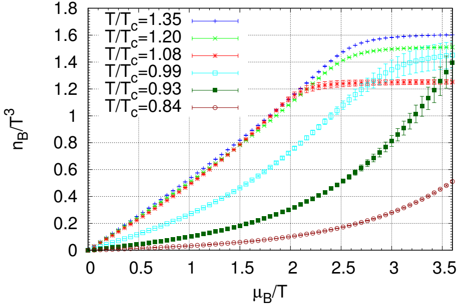
Extracted fugacity $\xi (= e^{\mu/T})$ agreed with HRG model estimation

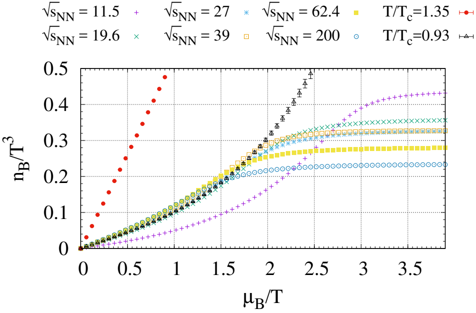
$\sqrt{s_{NN}}$, GeV	J. Cleynams	P. Alba	A. Nakamura
11.5	8.04	11.1	7.48
19.6	3.62	3.65	3.21
27.0	2.62	2.58	2.43
39.0	1.98	1.93	1.88
62.4	1.55	1.55	1.53
200.0	1.18	1.18	1.18

J. Cleaymans et al., Phys. Rev. C 73, 034905 (2006)
P. Alba et al., Phys. Let. B 738, 305 (2014)

Comparisson with RHIC experiment: Baryon density

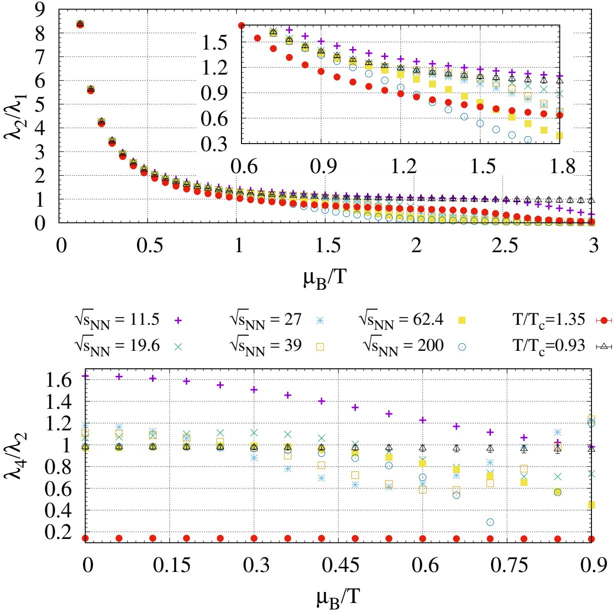
$$Z_{GC}(\mu) = \sum_{n=-\infty}^{\infty} Z_C(n) e^{\frac{\mu n}{T}} \quad \Rightarrow n_B = \frac{1}{V}\frac{\partial(ln Z_{GC})}{\partial(\mu/T)} = C\frac{2\sum_n^{n_{max}} n Z_n \sinh(n\mu/T)}{1 + 2\sum_n^{n_{max}} Z_n \cosh(n\mu/T)}$$





Experimental data are extracted from RHIC STAR **(Luo X. CEJP 10, 1372 (2012))**
(A. Nakamura, K. Nagata PTEP, 033D01 (2016))

Comparisson with RHIC experiment: Higher Moments



$$\lambda_n(\mu/T) = (T\frac{\partial}{\partial\mu})^n \log Z_{GC}(\mu/T)$$

$T_c = (154 \pm 9)$ МэВ
LQCD (arXive:1504.05274)

T and μ
HRG arXiv:1403.4903

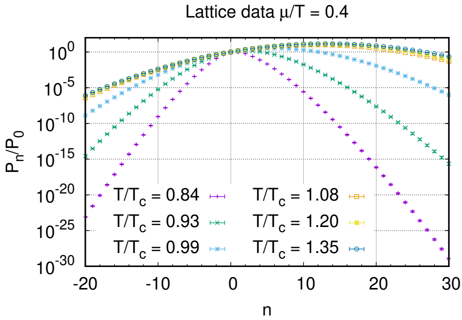
$\sqrt{s_{NN}}$, ГэВ	T/T_c
11.5	0.88
19.6	0.96
27	0.96
39	0.98
62.4	0.97
200	0.95

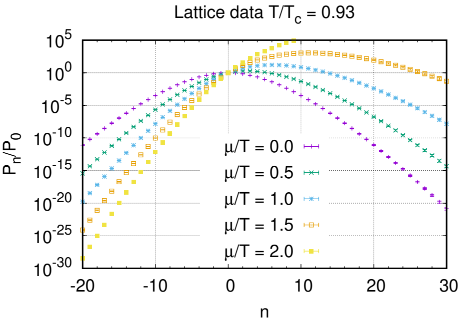
Multiplicity: Lattice data

Probability interpretation:

$$1 = \sum_n \frac{Z_n \xi^n}{Z_{GC}(\mu)}, \quad \xi = e^{\mu/T} \qquad \frac{N(n)}{N(0)} = \frac{N\, Z_n \xi^n / Z_{GC}(\mu)}{N\, Z_0 \xi^0 / Z_{GC}(\mu)} = Z_n \xi^n / Z_0$$

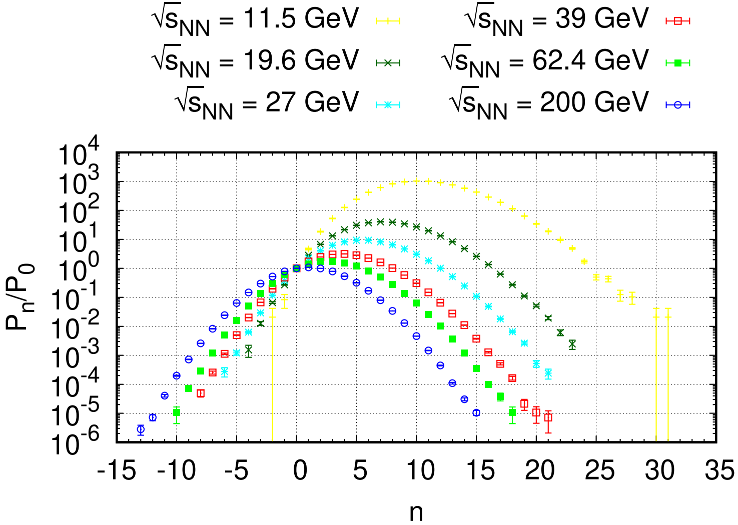
$$\Rightarrow \text{Multiplicity: } \frac{P_n}{P_0} = \frac{Z_n}{Z_0}\, e^{n\mu/T}$$



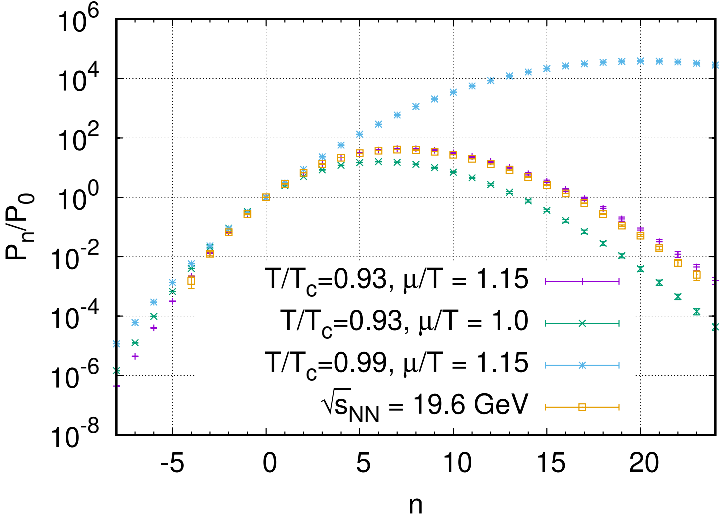


Multiplicity: RHIC experiment data

RHIC STAR data **(Luo X. CEJP 10, 1372 (2012))**

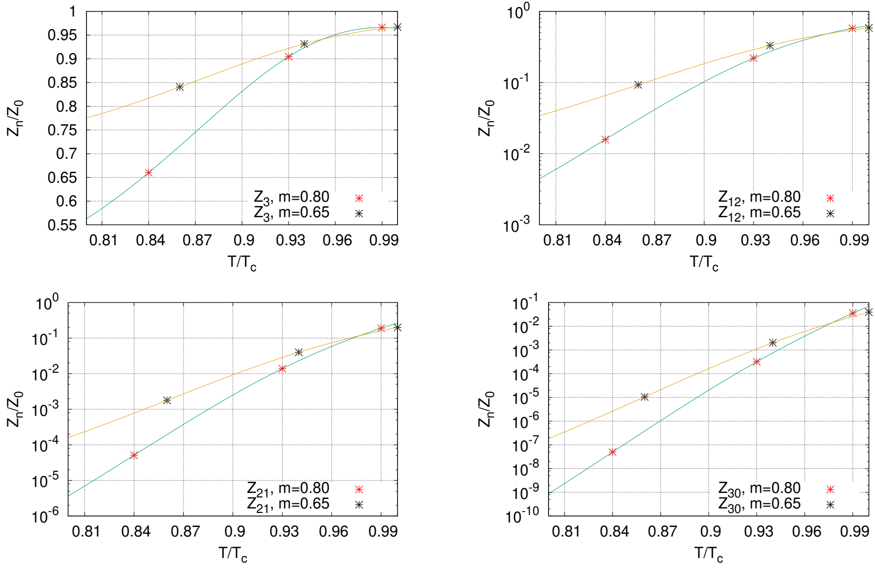


Multiplicity: fit Lattice data to RHIC data

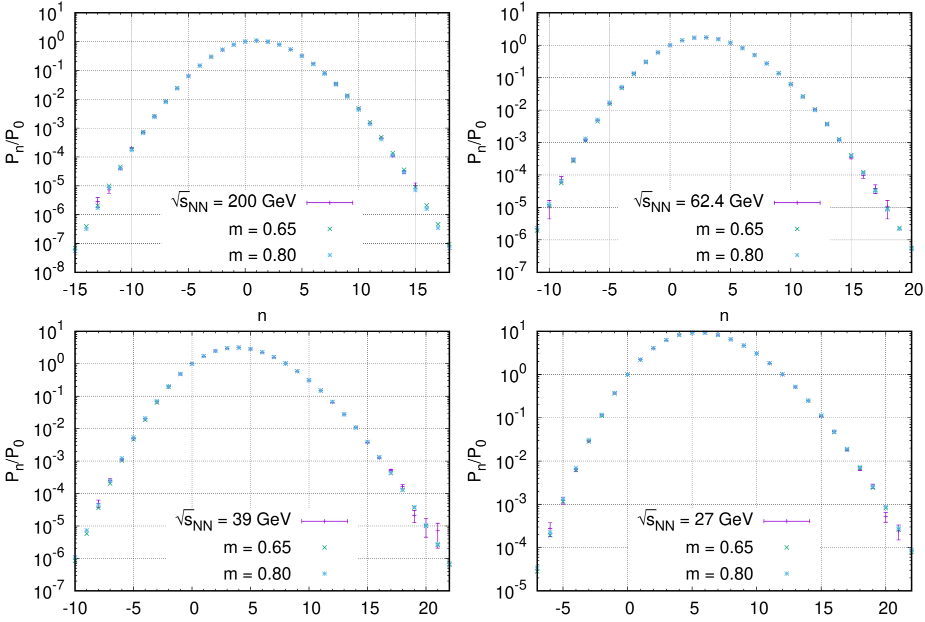


Multiplicity: interpolation on temperature

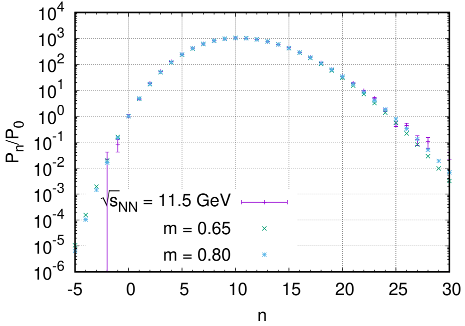
$$\frac{P_n}{P_0}(T,\mu) = \frac{Z_n}{Z_0}(T)\, e^{n\mu/T}$$

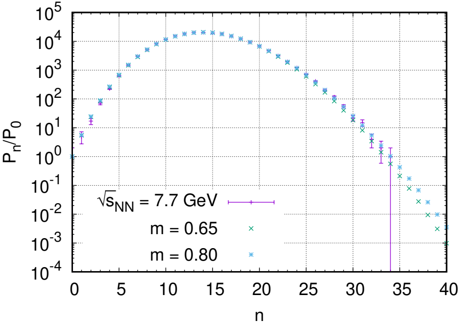


Multiplicity: RHIC experiment data and Lattice



Multiplicity: RHIC experiment data and Lattice (2)





Multiplicity: RHIC experiment parameters

(using Lattice data at $m_{pi}/m_\rho = 0.80$)

Temperature T/T_c

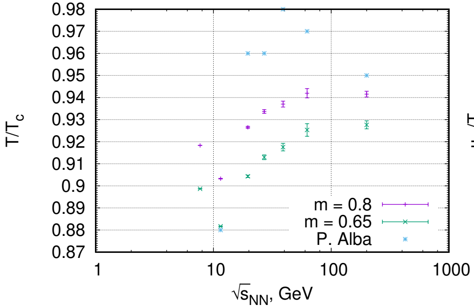
$\sqrt{s_{NN}}$, GeV	T/T_c P. Alba	T/T_c this work
7.7	-	0.918(0.03)(50)
11.5	0.88	0.903(0.13)(50)
19.6	0.96	0.926(0.44)(50)
27.0	0.96	0.934(0.84)(50)
39.0	0.98	0.937(1.32)(50)
62.4	0.97	0.942(2.11)(50)
200.0	0.95	0.942(1.31)(50)

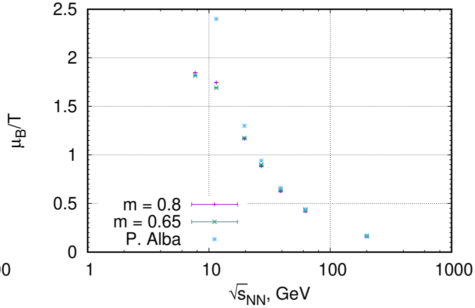
Chemical Potential μ_B/T

$\sqrt{s_{NN}}$, GeV	μ_B/T P. Alba	μ_B/T this work
7.7	-	1.846(0.3)
11.5	2.4	1.745(1.3)
19.6	1.30	1.167(4)
27.0	0.94	0.885(6)
39.0	0.66	0.627(9)
62.4	0.44	0.423(12)
200.0	0.17	0.161(6)

P. Alba et al., Phys. Let. B 738, 305 (2014)

Multiplicity: RHIC experiment parameters (2)





P. Alba et al., Phys. Let. B 738, 305 (2014)

Talk summary

- Canonical coefficients can be calculated with integration method: dependence of observable on imaginary chemical potential is required
- Experimental data can be analyses using Canonical Approach: good agreement with previous estimations is reached