# Energy-momentum tensor on the lattice and its applications

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References:

FlowQCD coll. Phys.Rev. D90 (2014) 1, 011501

E.I., H.Suzuki, Y.Taniguchi, T.Umeda arXiv:1511.03009

and

Work in progress with S. Aoki

Seminar @ Far Eastern Federal University, Russia 2016/04/07

# Energy-momentum tensor (EMT)

A fundamental quantity in quantum field theory

$$T_\mu^\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta_\mu^\nu \mathcal{L}$$

- generator of general coord. transformation
- conserved quantity (energy density, momentum, pressure)
- universal quantity (central charge in conformal theory)

$T^{00}$ energy density

$T^{0j}$ energy flux in j-dir.

$T^{i0}$ momentum (i-component)

$T^{ij}$ mom. flux in j-dir (i-com.)

# EMT on Lattice

Lattice regularization: a nonperturbative regularization

gauge invariant

discretize space-time coord.

- generator of general coord. transformation

# EMT on Lattice

Lattice regularization: a nonperturbative regularization

gauge invariant

discretize space-time coord.

- ✦ generator of general coord. transformation
- ✦ same quantum number with the vac. (signal is noisy)

# plan to talk

- ✦ Introduction…basic idea to obtain the EMT using the lattice
- ✦ One-point fn. of EMT in finite temperature quenched QCD
- ✦ two-point fn. of EMT
- ✦ (2+1)flavor QCD case
- ✦ Summary

# Basic Idea

Quantum field theory

(UV divergence)

perturbation with
dim. reg.
+YM gradient flow
(general covariance OK!)

lattice reg.
+Wilson flow
(with a->0 limit)

At finite flow time, UV finite!

Luescher and Weisz, JHEP 1102, 051(2011)

Firstly, we obtain the relation between them perturbatively.
Assume that it applies to the nonperturbative regime.

# YM gradient flow

## Flow equation

Luescher, JHEP 1008, 071 (2010)

Yang–Mills gradient flow (continuum theory)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots\,, \qquad B_\mu(t=0,x) = A_\mu(x)$$

Wilson flow (lattice theory)

$$\partial_t V(t,x,\mu) V(t,x,\mu)^{-1} = -g_0^2 \partial S_{\mathrm{Wilson}}\,, \qquad V(t=0,x,\mu) = U(x,\mu)$$

link variable: $U_\mu(x) = e^{ig_0 A_\mu(x)}$

t: fictitious time direction (flow-time)

$x = (\vec{x}, \tau)$

# UV finiteness of the gradient flow

## Flow equation (continuum)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x)$$ initial condition: $B_\mu(t=0,x) = A_\mu(x)$

## perturbative solution in the leading order

$$B_\mu(t,x) = \int d^D y K_t(x-y) A_\mu(y)$$

$$K_t(z) = \int \frac{d^D p}{(2\pi)^D} e^{ipz} e^{-tp^2}$$

$p^2 > 1/t$ modes are suppressed (a smooth UV cutoff)

Smeared in the range $|x| < \sqrt{8t}$

signal becomes clear?

Finiteness is shown perturbatively in all order

Luescher and Weisz, JHEP 1102, 051(2011)

# Energy-momentum tensor

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x)\right]$$

Renormalized EMT within dim. reg.

$$\{T_{\mu\nu}\}_R(x) = T_{\mu\nu}(x) - \langle T_{\mu\nu}(x)\rangle$$

Dim=4 gauge invariant operator on Lattice

$$U_{\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) - \frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x)$$

$$E(t,x) \equiv \frac{1}{4}G^a_{\mu\nu}(t,x)G^a_{\mu\nu}(t,x)$$

Here, ops. are constructed by flowed field.

# ``Suzuki method"

## - small flow-time expansion -

Suzuki, PTEP 2013, no8, 083B03, [Erratum: PTEP2015,079201(2015)],

relation… dim.=4 op on the lattice vs. renormalized EMT at small flow-time

$$U_{\mu\nu}(t,x) = \alpha_U(t)\left[\{T_{\mu\nu}\}_R(x) - \frac{1}{4}\delta_{\mu\nu}\{T_{\rho\rho}\}_R(x)\right] + O(t),$$

$$E(t,x) = \langle E(t,x)\rangle + \alpha_E(t)\{T_{\rho\rho}\}_R(x) + O(t),$$

coefficients… given by renormalized coupling and coeff. of beta fn.

$$\alpha_U(t)(g;\mu) = g^2\left\{1 + 2b_0\left[\ln(\sqrt{8t}\mu) + s_1\right]g^2 + O(g^4)\right\},$$

$$\alpha_E(t)(g;\mu) = \frac{1}{2b_0}\left\{1 + 2b_0 s_2 g^2 + O(g^4)\right\},$$

$b_0$ 1-loop coeff. of beta fn. MSbar scheme

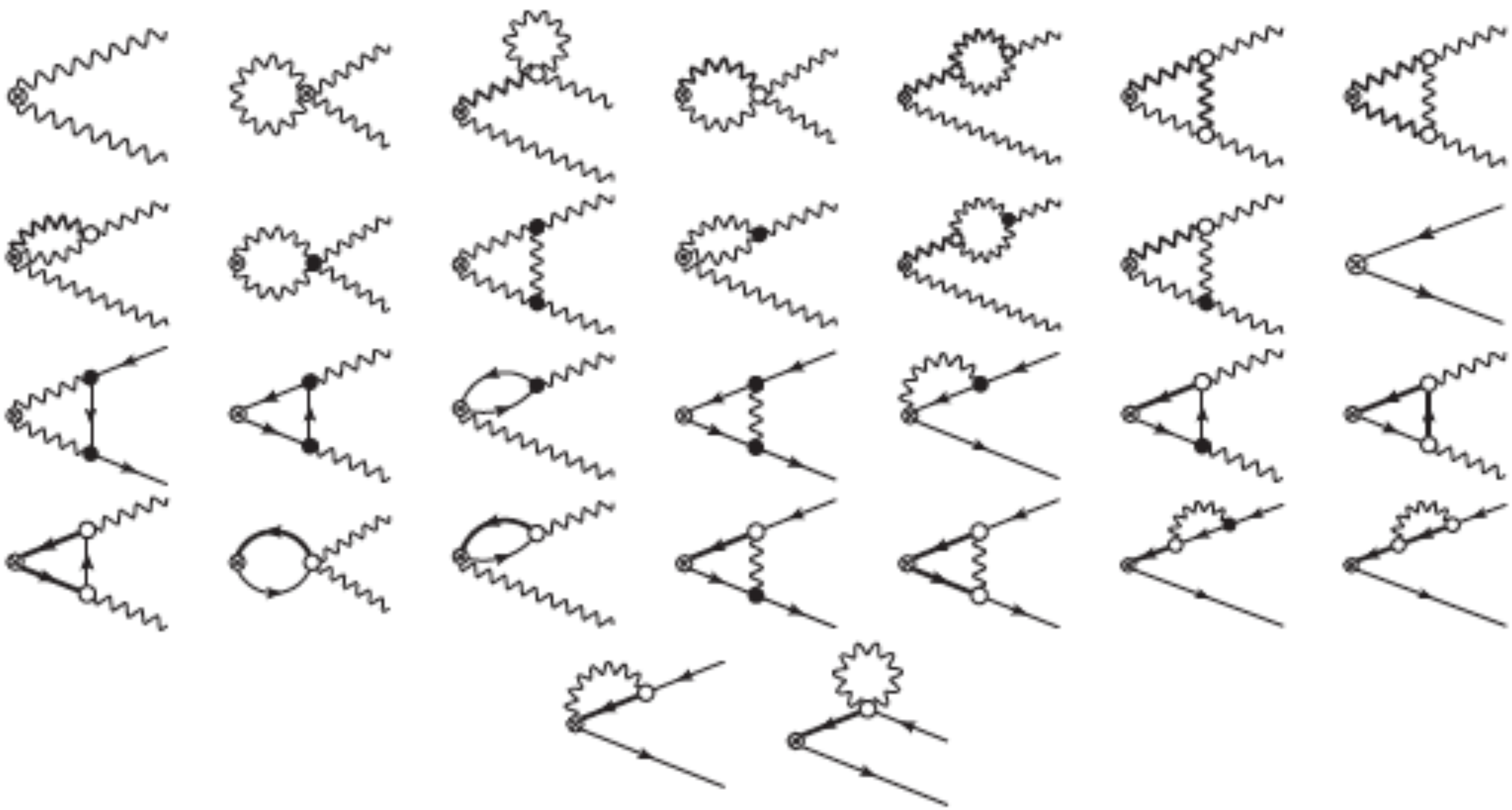
$s_1 =$ -0.0863575

$s_2 =$ 0.05578512

cf.) Nonperturbative method: L.DelDebbio, A.Patella,A.Rogo, JHEP 1311,212(2013)

# Calculation of $\zeta_{ij}(t)$

- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



Suzuki-san's slide @ HHIQCD2015,Kyoto

# How to get EMT

for quenched QCD

## Step 1

Generate gauge configuration at t=0 (usual process)

## Step 2

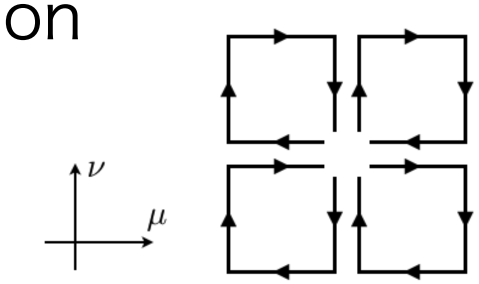
Solve the Wilson flow eq. and generate the gauge configuration at flow time (t)

$$a \ll \sqrt{8t} \ll \Lambda_{QCD}^{-1} \text{ or } T^{-1}$$

## Step 3

Measure two dim=4 ops. using flowed gauge configuration

$$U_{\mu\nu}(t,x), E(t,x)$$




## Step 4

Take the continuum limit. Then take t->0 limit.
(Take care the feasible window of flow time)

$$T^R_{\mu\nu}(x) = \lim_{t\to 0}\left\{\frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)}[E(t,x) - \langle E(t,x)\rangle_0]\right\}$$

# One-point fn. of EMT in finite temperature quenched QCD

Asakawa, Hatsuda, E.I., Kitazawa, Suzuki (FlowQCD coll.)

Phys.Rev. D90 (2014) 1, 011501

# Simulation setup

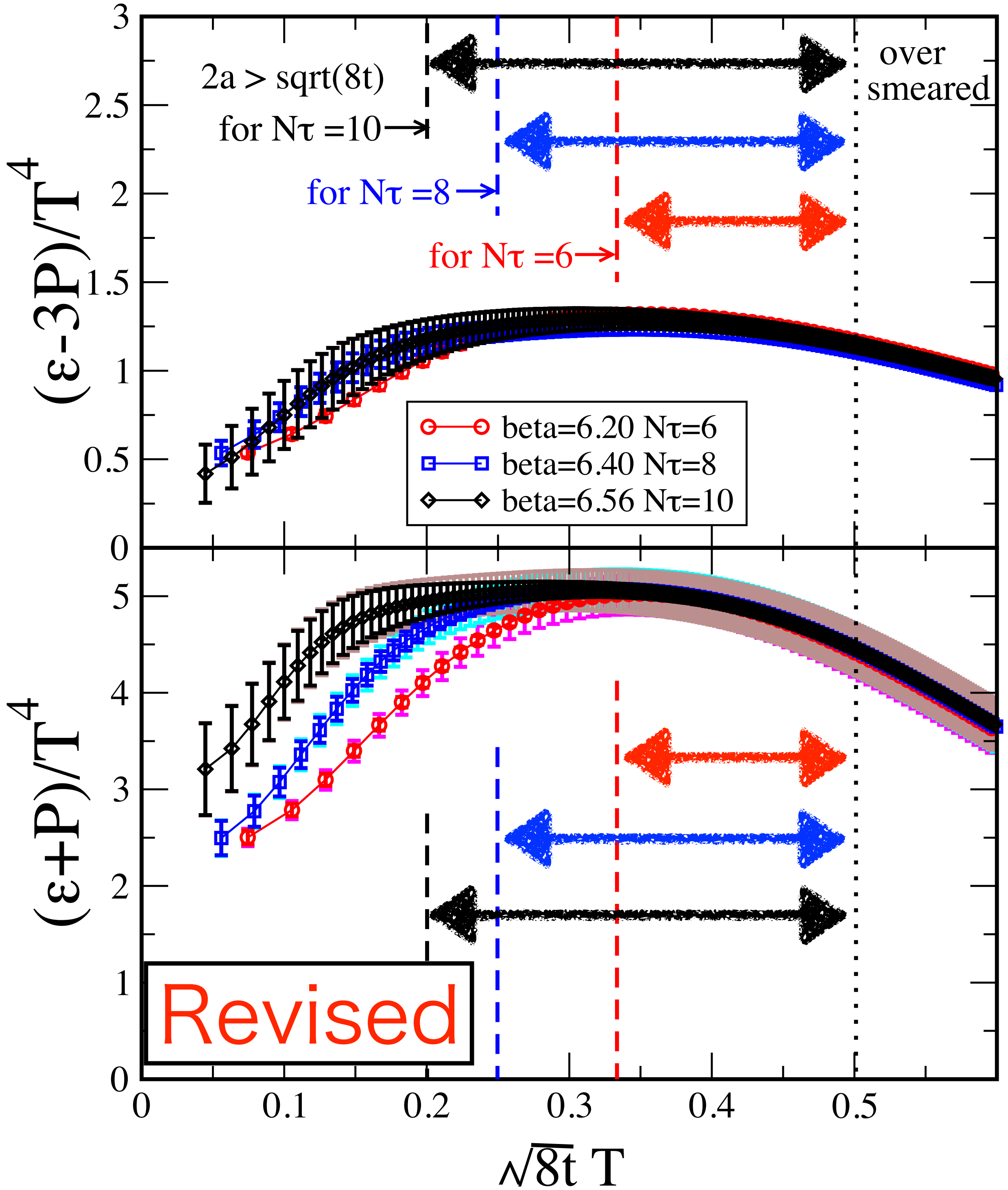
- Wilson plaquette gauge action
- lattice size (Ns=32, Nt=6,8,10,32)
- # of confs. is 100 - 300
- simulation parameters

| $N_\tau$ | 6 | 8 | 10 | $T/T_c$ |
|---|---|---|---|---|
| $\beta$ | 6.20 | 6.40 | 6.56 | 1.65 |
| | 6.02 | 6.20 | 6.36 | 1.24 |
| | 5.89 | 6.06 | 6.20 | 0.99 |

Temperature is determined by
Boyd et. al. NPB469,419 (1996)

Parametrization is given by
alpha collaboration NPB538,669 (1999)

# flow time dependence (T=1.65Tc)




each dark color shows statistical error

each light color includes systematic error

trace anomaly $\sum_{i=1}^{4} T_{ii} = \dfrac{\epsilon - 3P}{T^4}$

entropy density $T_{44} - T_{11} = \dfrac{\epsilon + P}{T^4}$

## feasible flow time

longer than lattice cutoff

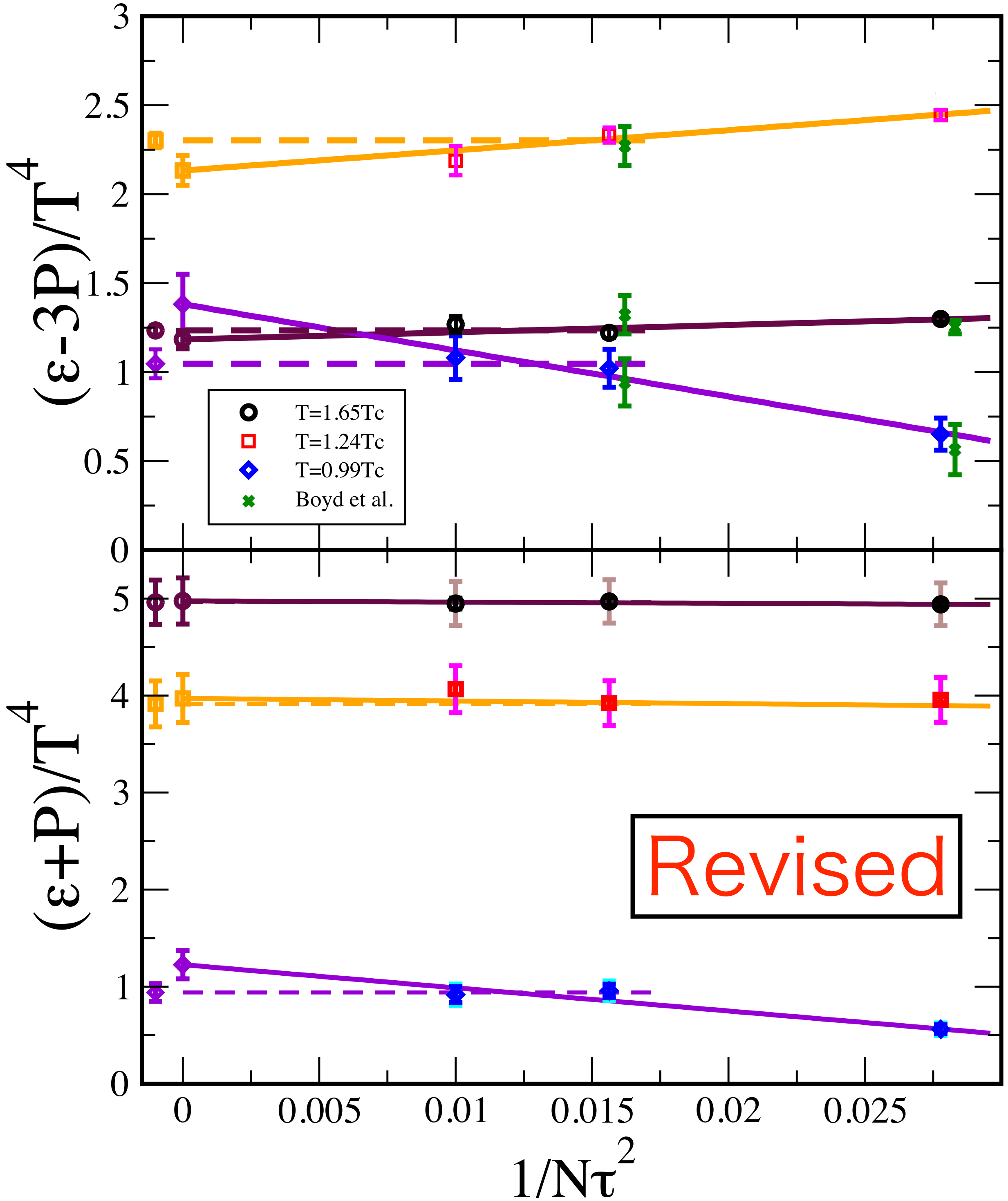
avoid an over-smeared regime

$$2a < \sqrt{8t} < N_\tau a/2$$

- show a plateau
  (small higher dimensional op.)
  Practically, no need t-> 0 limit
  **finer lattice simulation shows a slope

- systematic error coming from scale setting is dominated in entropy density

# Continuum extrapolation




$\sqrt{8t}T = 0.40$

3point linear extrap.
(2pt. const. extrap.)

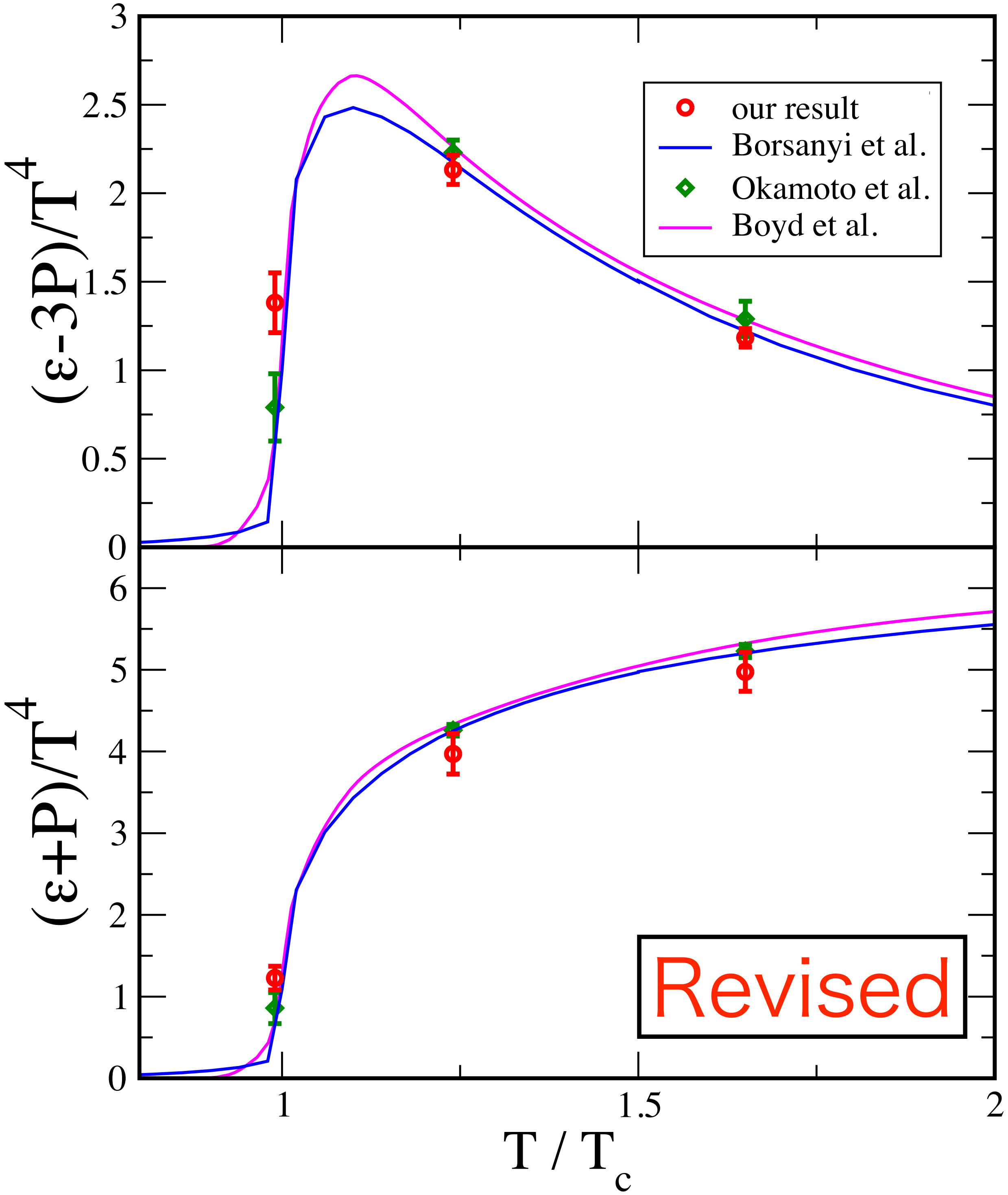
We also see the data at $\sqrt{8t}T = 0.35$

In cont.lim. the result is consistent.

t->0 limit is not needed in this case

# Comparison with the results given by integration method

Phys.Rev. D90 (2014) 1, 011501, arXiv:1312.7492v3[hep-lat]




Boyd et. al. NPB469,419 (1996)

Okamoto et. al. (CP-PACS) PRD60, 094510 (1999)

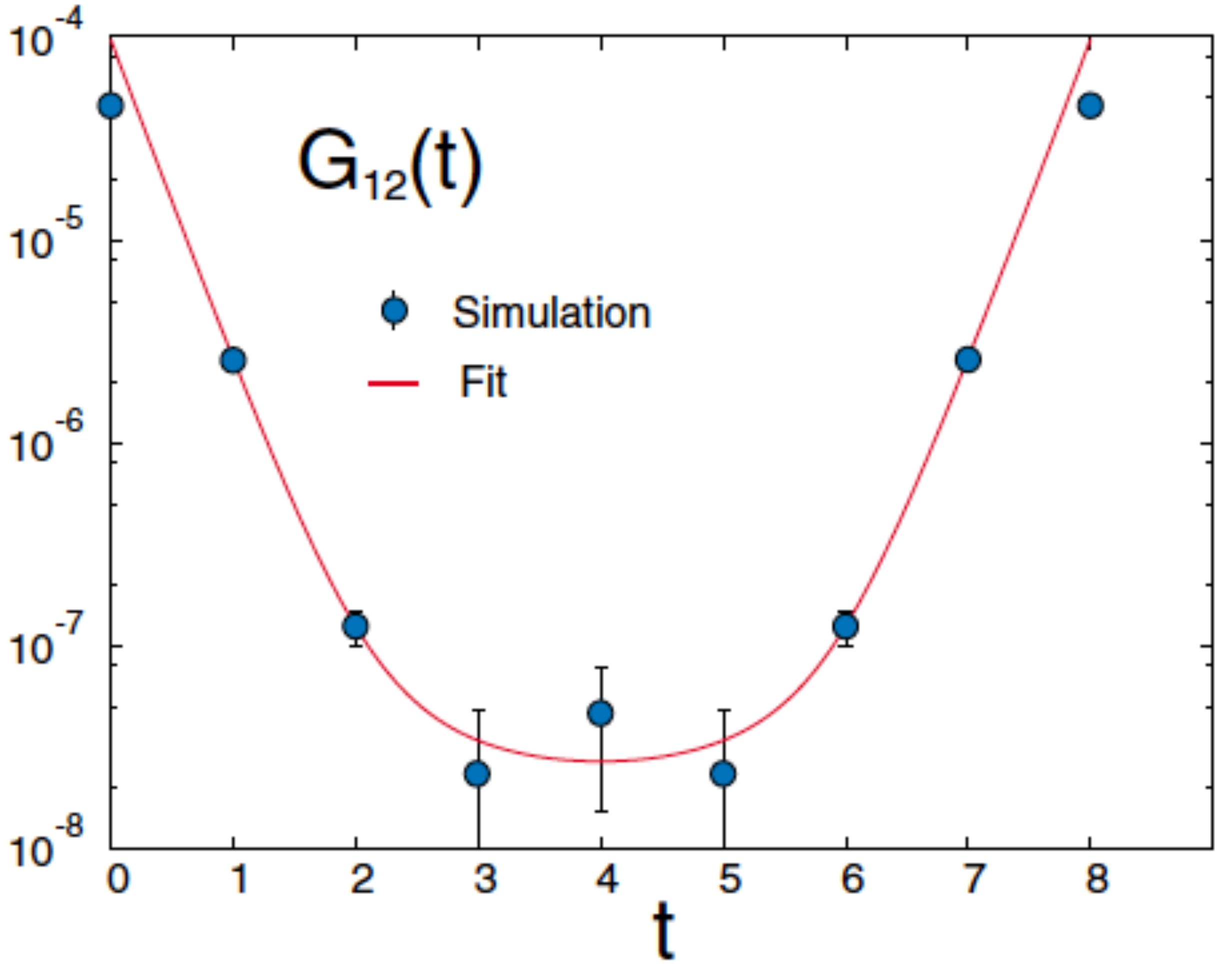
Borsanyi et. al. JHEP 1207, 056 (2012)

Integration method is based on (macroscopic) thermodynamics.

Our method is based on the (microscopic) quantum field theory.

two-point fn. of EMT

# Shear viscosity in QGP phase




Matsubara-Green's function G12(t),
Nakamura-Sakai(2005)
800,000 conf.

shear viscosity: retarded Green's fn.

$$\eta = -\int \langle T_{12}(\vec{x},\tau) T_{12}(\vec{x}',0)\rangle_{\mathrm{ret.}}$$

obtained by the analytic continuation of Matsubara Green's fn.

$$G_\beta(\vec{p},t) = \sum_n e^{i\omega_n t} \int d\omega \frac{\rho(\vec{p},\omega)}{i\omega_n - \omega}$$

# Renormalization

$$\boxed{T_{\mu\nu}^{(R)}(g_0) = Z(g_0)T_{\mu\nu}^{(bare)}}$$

Meyer (2007)…1loop approximation

Fodor et al. (2013)…calculate Z-factor from entropy density

This work … Not necessary

(usage of Suzuki coefficient and MSbar coupling)

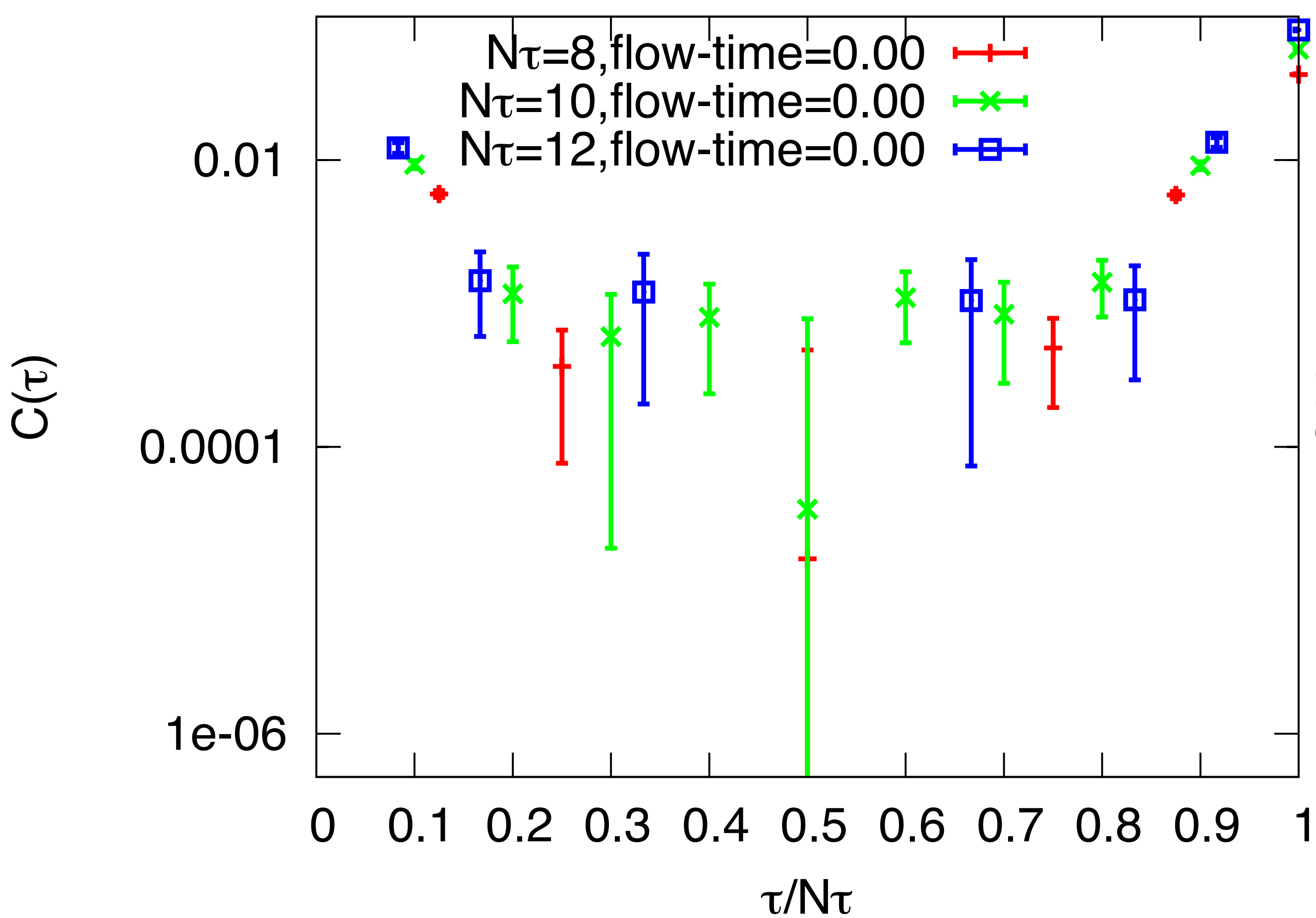
cf.) $\frac{sT}{4} = \langle T_{11}^{(R)} \rangle$

$$\langle T_{12}T_{12} \rangle = \frac{1}{4}\langle (T_{11} - T_{22})(T_{11} - T_{22}) \rangle$$

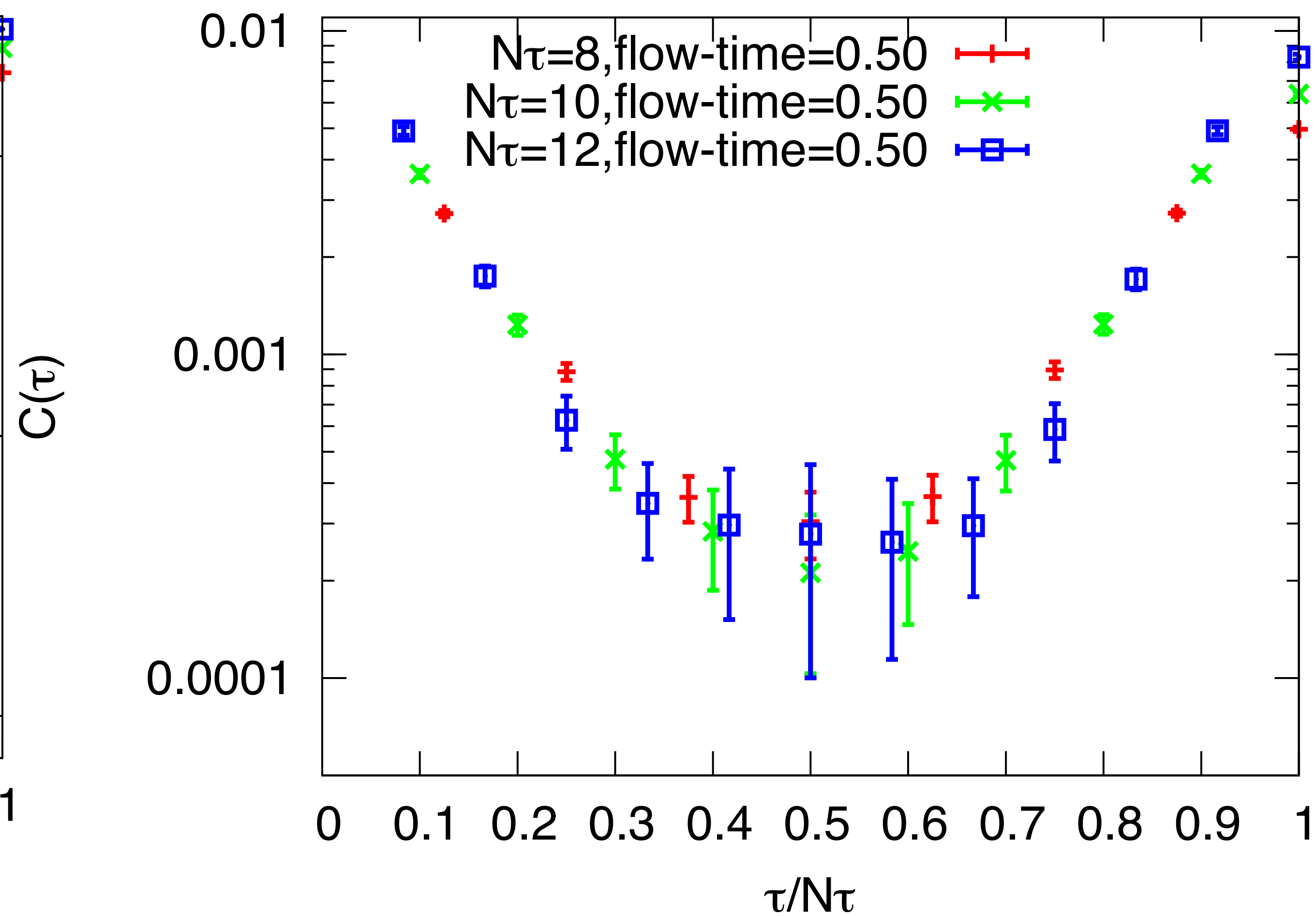
# lattice raw data

$$C(\tau) = \langle \frac{1}{N_s^3} \sum_{\vec{x}} U_{12}(\vec{x},\tau) \frac{1}{N_s^3} \sum_{\vec{y}} U_{12}(\vec{y},0) \rangle$$

flow-time=0




flow-time t/a^2=0.50




fixed smeared length in lattice unit

beta=6.40,Nt=8, 2,000 conf.

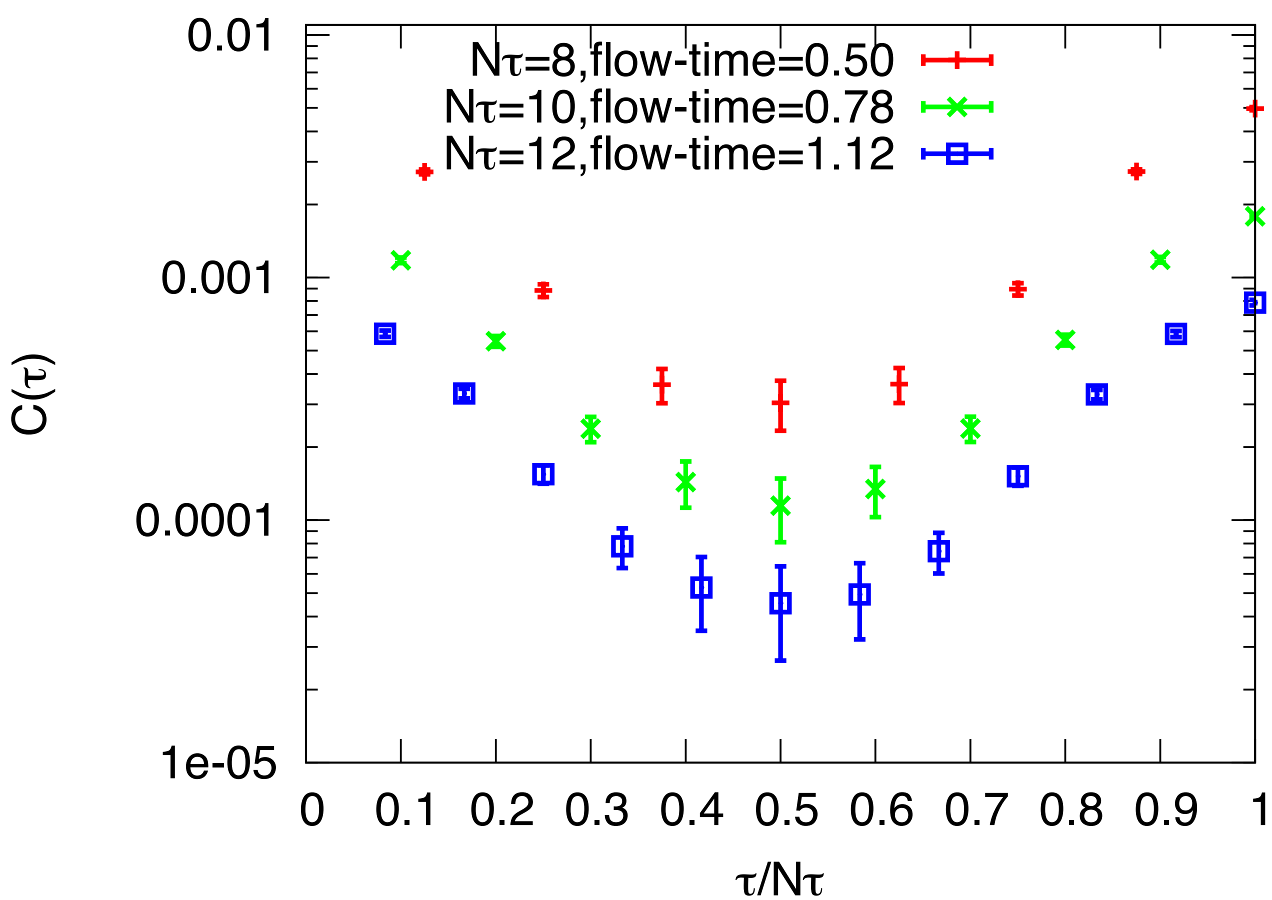
beta=6.57,Nt=10, 1,100 conf.

beta=6.72,Nt=12, 650 conf.

# EMT correlator

$$T^R_{\mu\nu}(x) = \lim_{t\to 0}\left\{\frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)}[E(t,x) - \langle E(t,x)\rangle_0]\right\}$$

$$C(\tau) = \langle \frac{1}{N_s^3}\sum_{\vec{x}} U_{12}(\vec{x},\tau)\frac{1}{N_s^3}\sum_{\vec{y}} U_{12}(\vec{y},0)\rangle$$

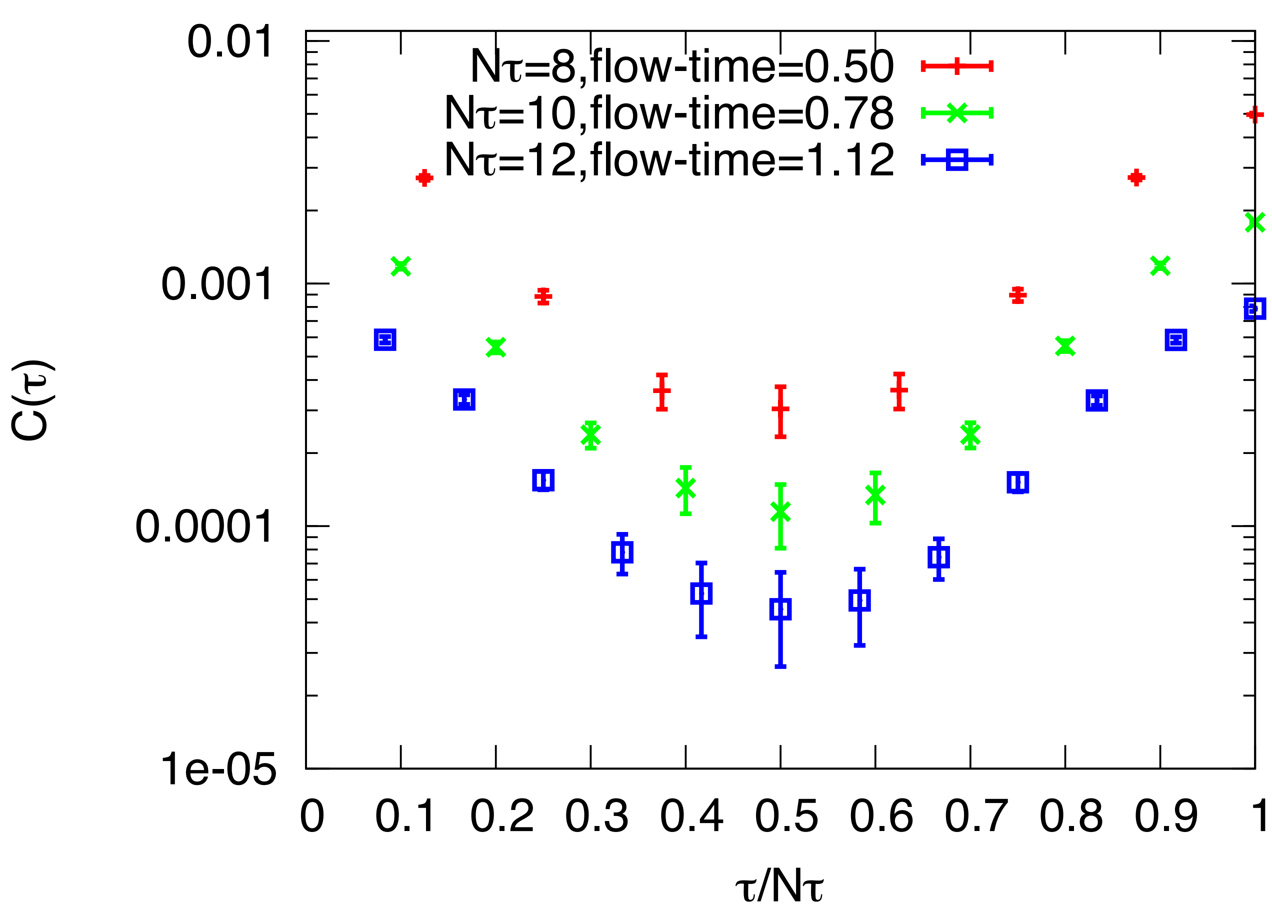



fixed smeared length in physical unit $\sqrt{8t}T = 0.25$

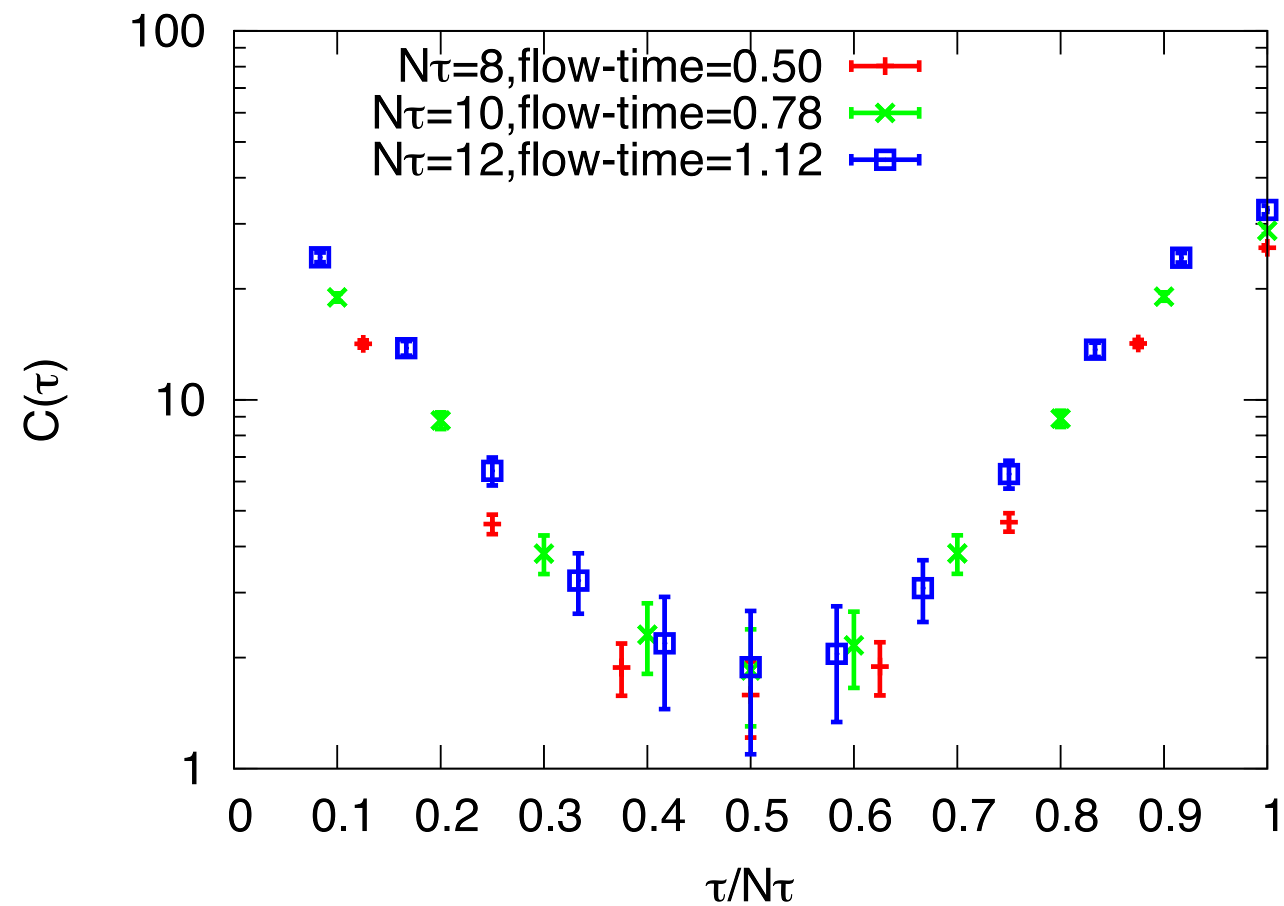
# EMT correlator

$$T^R_{\mu\nu}(x) = \lim_{t\to 0}\left\{\frac{1}{\alpha_U(t)}U_{\mu\nu}(t,x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)}[E(t,x) - \langle E(t,x)\rangle_0]\right\}$$

$$C(\tau) = \langle \frac{1}{N_s^3}\sum_{\vec{x}} U_{12}(\vec{x},\tau)\frac{1}{N_s^3}\sum_{\vec{y}} U_{12}(\vec{y},0)\rangle$$




$$C(\tau) = \frac{1}{T^5}\langle \sum_{\vec{x}} T_{12}(\vec{x},\tau)\sum_{\vec{y}} T_{12}(\vec{y},0)\rangle$$




fixed smeared length in physical unit $\sqrt{8t}T = 0.25$

# I am doing…

- ✦ accumulate the data
- ✦ try to fit the two-point fn. to obtain the shear viscosity…

# (2+1)flavor QCD

E.I., H.Suzuki, Y.Taniguchi, T.Umeda arXiv:1511.03009

# How to get EMT

## for quenched QCD

## Step 1

Generate gauge configuration at t=0 (usual process)

## Step 2

Solve the Wilson flow eq. and generate the gauge configuration at flow time (t)

$$a \ll \sqrt{8t} \ll \Lambda_{QCD}^{-1} \text{ or } T^{-1}$$

## Step 3

Measure two dim=4 ops. using flowed gauge configuration

$$U_{\mu\nu}(t,x), E(t,x)$$

## Step 4

Take the continuum limit. Then take t->0 limit.
(Take care the feasible flow time window)

$$T_{\mu\nu}^{R}(x) = \lim_{t\to 0}\left\{ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t,x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)}[E(t,x) - \langle E(t,x)\rangle_0] \right\}$$

# How to get EMT

for full QCD

## Step 1

Generate gauge configuration at t=0 (usual process)

With fermion

## Step 2

Solve the Wilson flow eq. and generate the gauge configuration at flow time (t)

$$a \ll \sqrt{8t} \ll \Lambda_{QCD}^{-1} \text{ or } T^{-1}$$

With fermion flow

## Step 3

M.Luescher, JHEP 04 (2013) 123

Measure two dim=4 ops. using flowed gauge configuration

$$U_{\mu\nu}(t,x), E(t,x)$$

Add operators with fermion

## Step 4

H.Makino and H.Suzuki, PTEP 2014 (2014) 6, 063B02

Take the continuum limit. Then take t->0 limit.
(Take care the feasible flow time window)

# operators with fermion

H.Makino and H.Suzuki, PTEP 2014 (2014) 6, 063B02

$$
\begin{aligned}
\{T_{\mu\nu}\}_R(x) = c_1(t)&\left[\tilde{O}_{1\mu\nu}(t,x) - \frac{1}{4}\tilde{O}_{2\mu\nu}(t,x)\right] \\
&+ c_2(t)\left[\tilde{O}_{2\mu\nu}(t,x) - \left\langle \tilde{O}_{2\mu\nu}(t,x)\right\rangle\right]
\end{aligned}
$$

added

$$
\left\{
\begin{aligned}
&+ c_3(t)\left[\tilde{O}_{3\mu\nu}(t,x) - 2\tilde{O}_{4\mu\nu}(t,x) - \left\langle \tilde{O}_{3\mu\nu}(t,x) - 2\tilde{O}_{4\mu\nu}(t,x)\right\rangle\right] \\
&+ c_4(t)\left[\tilde{O}_{4\mu\nu}(t,x) - \left\langle \tilde{O}_{4\mu\nu}(t,x)\right\rangle\right] \\
&+ c_5(t)\left[\tilde{O}_{5\mu\nu}(t,x) - \left\langle \tilde{O}_{5\mu\nu}(t,x)\right\rangle\right] + O(t),
\end{aligned}
\right.
$$

essentially summarized

$$
\begin{aligned}
O^f_{3\mu\nu}(t,x) &\equiv \varphi^f(t)\bar{\chi}^f(t,x)\left(\gamma_\nu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\chi^f(t,x), \\
O^f_{4\mu\nu}(t,x) &\equiv \varphi^f(t)\delta_{\mu\nu}\bar{\chi}^f(t,x)\gamma_\rho \overleftrightarrow{D}_\rho \chi^f(t,x), \\
O^f_{5\mu\nu}(t,x) &\equiv \varphi^f(t)\delta_{\mu\nu}\bar{\chi}^f(t,x)\chi^f(t,x),
\end{aligned}
$$

$$
\begin{aligned}
t^r_{\mu\nu}(t) &\equiv \frac{1}{N_\Gamma}\sum_x \left\langle \bar{\chi}_r(t,x)\gamma_\mu \left(D_\nu - \overleftarrow{D}_\nu\right)\chi_r(t,x)\right\rangle, \\
s^r(t) &\equiv \frac{1}{N_\Gamma}\sum_x \left\langle \bar{\chi}_r(t,x)\chi_r(t,x)\right\rangle,
\end{aligned}
$$

# Solve the fermion flow

M.Luescher, JHEP 04 (2013) 123

positive flow / adjoint flow

t

**Gauge flow**

$$\partial_t V_t = Z(V_t)V_t,$$

**Fermion (adjoint) flow**

$$\partial_t \chi_t = \Delta(V_t)\chi_t,$$

initial cond. $\xi_t^\epsilon(x) = \eta(x),$

Runge-Kutta step

$$W_3 = \exp\{\tfrac{3}{4}Z_2 - \tfrac{8}{9}Z_1 + \tfrac{17}{36}Z_0\}W_2,$$

$$W_2 = \exp\{\tfrac{8}{9}Z_1 - \tfrac{17}{36}Z_0\}W_1,$$

$$W_1 = \exp\{\tfrac{1}{4}Z_0\}W_0,$$

$$W_0 = V_t,$$

Runge-Kutta step

$$\lambda_3 = \xi_{s+\epsilon}^\epsilon,$$

$$\lambda_2 = \tfrac{3}{4}\Delta_2\lambda_3,$$

$$\lambda_1 = \lambda_3 + \tfrac{8}{9}\Delta_1\lambda_2,$$

$$\lambda_0 = \lambda_1 + \lambda_2 + \tfrac{1}{4}\Delta_0(\lambda_1 - \tfrac{8}{9}\lambda_2)$$

initial cond. $V|_{t=0} = U$

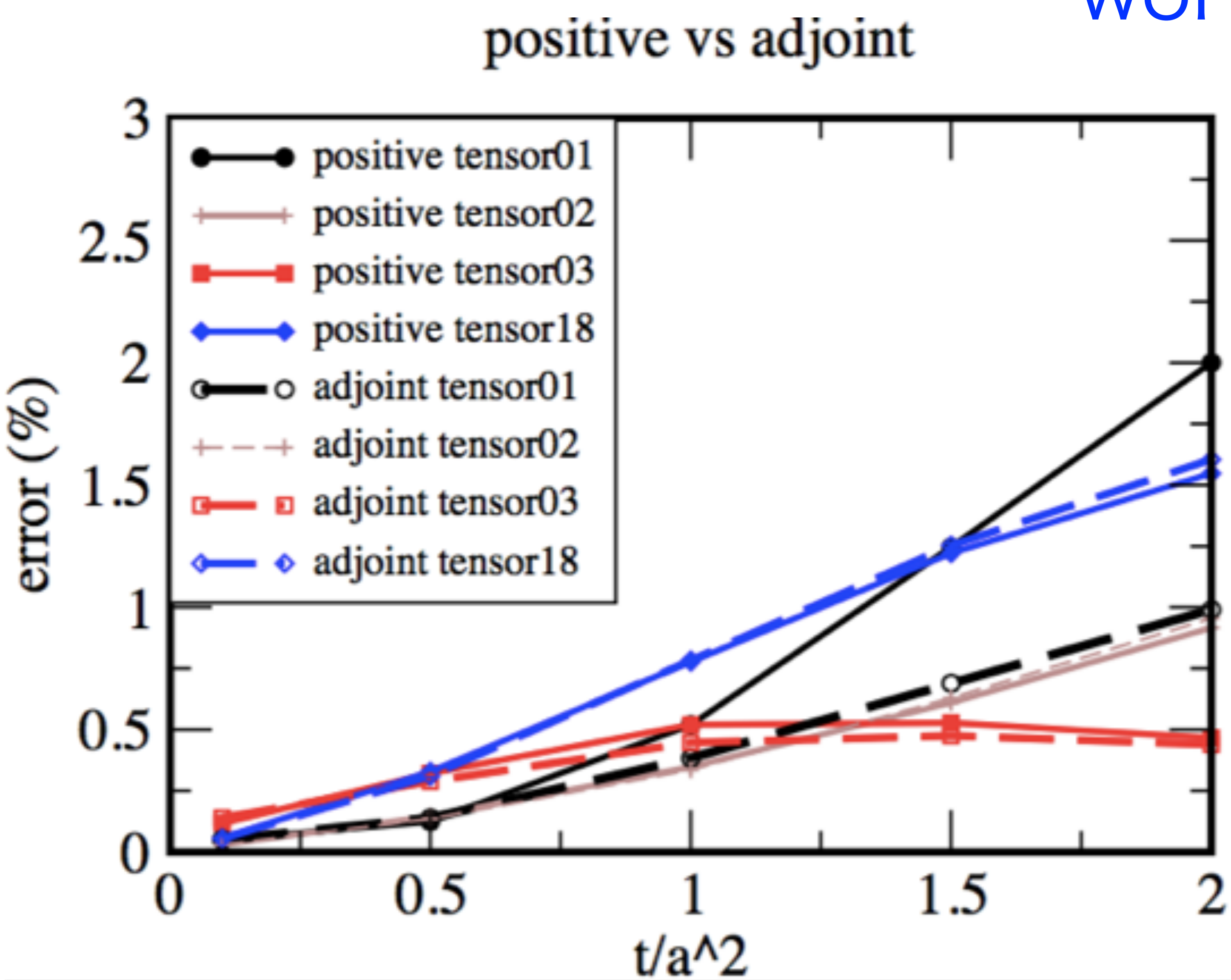
# Lattice setup

- Iwasaki gauge action + improved Wilson fermion
- lattice size (Ns=32, Nt=8)
- m_ps/m_v=0.6337(38) for u,d quarks
- m_ps/m_v=0.7377(28) for s quark
- each configuration is separated by 100 MC trj.

Parametrization is given by
T.Umeda et.al. for WHOT-QCD coll., Phys.Rev.D85,094508(2012)

# Statistical uncertainty

work in progress with S.Aoki




Solid data : positive flow

dashed data: adjoint flow

Same color indicates the same tensor component

tensor03 => t_{11}

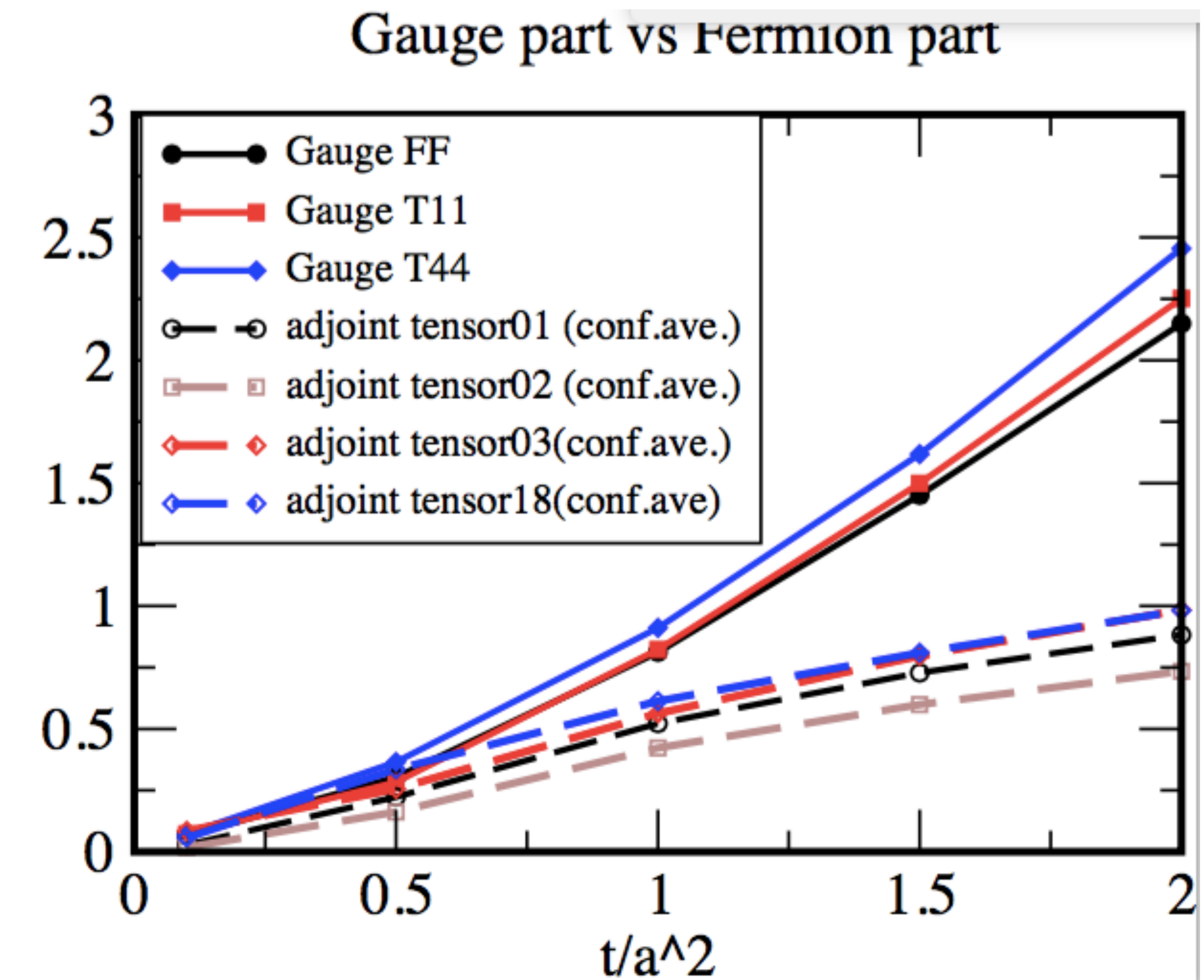
tensor18 => t_{44}

Most comp. have the same order of stat. err.

Numerical cost of positive flow is 10-times smaller than the one of adjoint flow.

# Statistical uncertainty

work in progress with S.Aoki




1,000 measurements for gauge ops.

100 measurements for fermion ops.

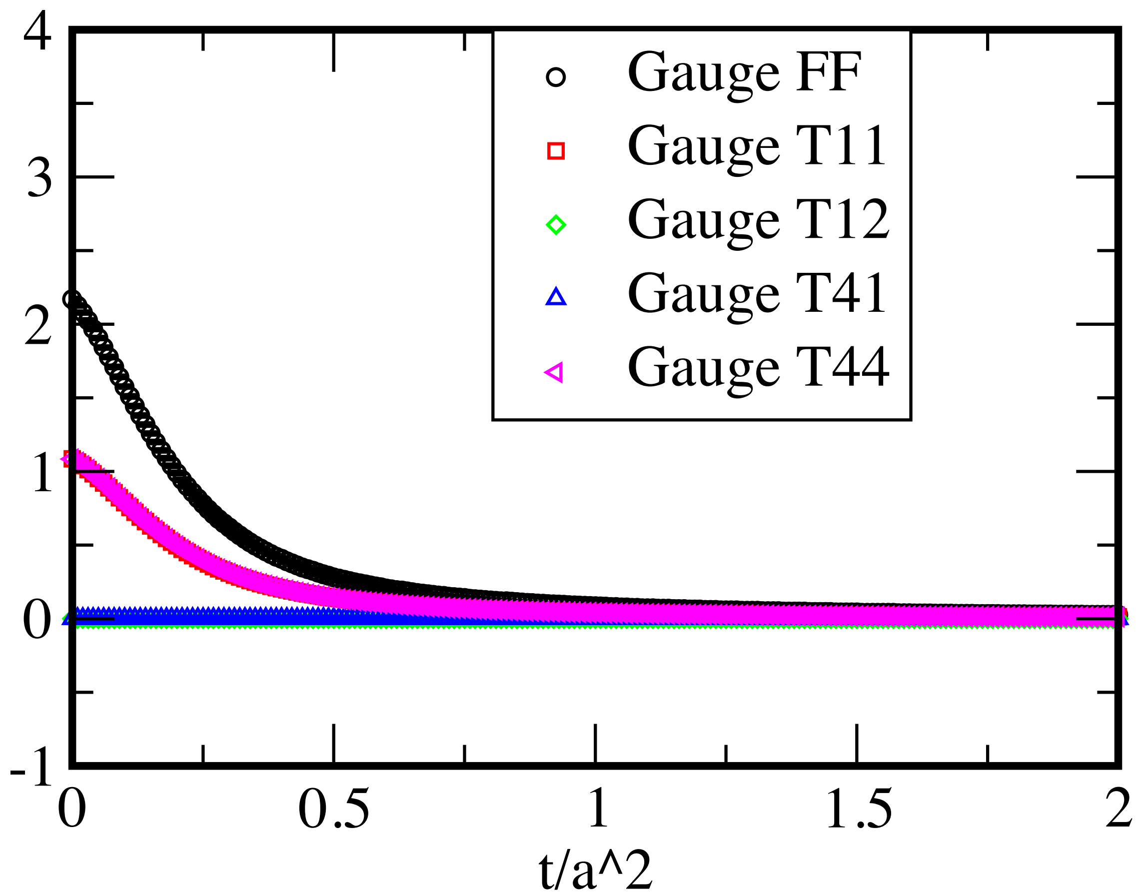
Gauge part has a large stat. err.

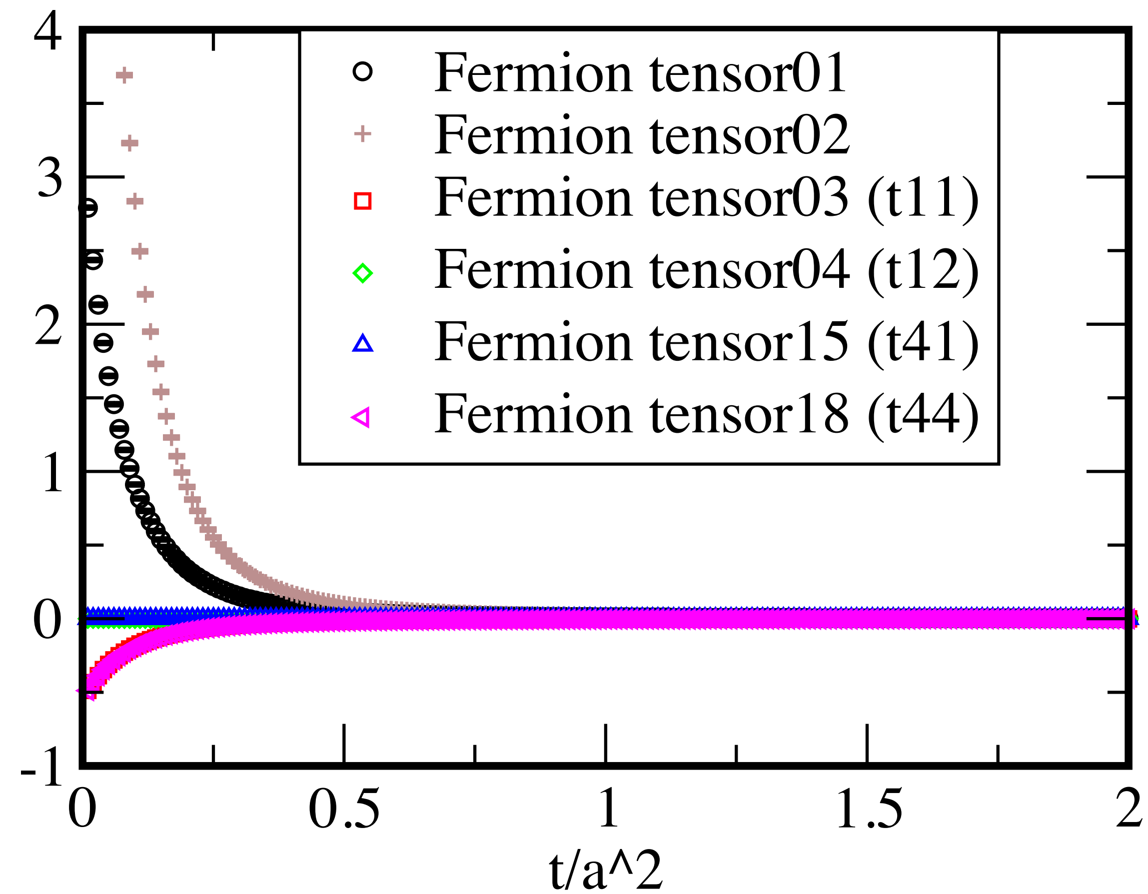
# preliminary results

## T=280MeV

Nt=8, beta=1.9728, c_sw=1.66922, kappa_ud=0.136147, kappa_s=0.135417

$$s^r(t) \equiv \frac{1}{N_\Gamma}\sum_x \langle \bar{\chi}_r(t,x)\chi_r(t,x)\rangle \qquad t^r_{\mu\nu}(t) \equiv \frac{1}{N_\Gamma}\sum_x \left\langle \bar{\chi}_r(t,x)\gamma_\mu\left(D_\nu - \overleftarrow{D}_\nu\right)\chi_r(t,x)\right\rangle$$




signal is very clear

# Summary of full QCD

- ✦To solve the fermion flow, the positive flow is better to obtain EMT tensor in the point of view of the simulation cost

- ✦ Gauge parts of EMT is noisy rather than fermion parts

## To do list

- finish the simulation
- calculate five Suzuki-coefficients
- combine 5 types operators

# conclusion

- ✦ Novel method to obtain EMT using the lattice simulation
- ✦ quenched results (1pt.fn) show that the small flow time expansion is promising
- ✦ clear statistical signal, small systematic error
- ✦ 2pt. fn. and full QCD simulation are also doable!!

# future directions

- ✦ two-point function of EMT (shear and bulk viscosity, heat capacity)
- ✦ application to conformal field theory (central charge, dilation physics)
- ✦ supersymmetry on the lattice
- ✦ nonlinear sigma model

# YM gradient flow

## Applications:

Luescher, (Lattice2013) arXiv:1308.5598

- topological charge
- chiral condensate
- scale setting (t0, w0)
- renormalized coupling
- energy-momentum tensor
- Holography (S.Aoki, K.Kikuchi, T.Onogi, arXiv:1505.00131)
- chiral gauge theory (D.M.Grabowska, D.B.Kaplan, arXiv:1511.03649)